On some new sharp embedding theorems for multifunctional Herz-type and Bergman-type spaces in tubular domains over symmetric cones

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1. Introduction. The goal of this paper to provide new sharp embedding theorems for new multi-functional analytic function spaces of several complex variables in typical unbounded Siegel domains of second type, namely in tubular domains over symmetric cones. Note various new sharp such type embedding results in context of onefunctional analytic Bergman, Hardy or Bergman type spaces appeared recently in several papers. We refer for such type results to [1–4], [20–24]. And also, to various references there. We also note that various embedding theorems have many applications in the study of actions of various integral operators in analytic function spaces and in the study of various related problems in analytic one function theory. We refer for these issues to [14–19] [26], [27], [31–35].

Note however multifunctional analytic function spaces and various sharp embedding theorems for them are less studied. This paper probably provides first such type sharp results in unbounded tubular domains over symmetric cones. Related results for new multifunctional analytic function spaces of several complex variables, but in context of bounded domains namely in some bounded strongly pseudoconvex domains with smooth boundary can be seen in recent paper of the first author [25].

Let $T_\Omega = V + i\Omega$ be the tube domain over an irreducible symmetric cone Ω in the complexification $V^{\mathbb{C}}$ of an n-dimensional Euclidean space V. $\mathcal{H}(T_\Omega)$ denotes the space of all holomorphic functions on T_Ω. Following the notation of [28] and [29] we denote the rank of the cone Ω by r and by Δ the determinant function on V.

Letting $V = \mathbb{R}^n$, we have as an example of a symmetric cone on $\mathbb{R}^n$ the Lorentz cone Λ_n which is a rank 2 cone defined for $n \geq 3$ by

$$\Lambda_n = \{y \in \mathbb{R}^n : y_1^2 - \cdots - y_n^2 > 0, y_1 > 0\}$$

The determinant function in this case is given by the Lorentz form

$$\Delta(y) = y_1^2 - \cdots - y_n^2.$$

(see for example [29]).

Also, if $t, k \in \mathbb{R}^r$, then $t < k$ means $t_j < k_j$ for all $1 \leq j \leq r$.

For $\tau \in \mathbb{R}_+$ and the associated determinant function $\Delta(x)$ [29] we set

$$A_\tau^\infty(T_\Omega) = \left\{ F \in \mathcal{H}(T_\Omega) : \|F\|_{A_\tau^\infty} = \sup_{x+iy \in T_\Omega} |F(x+iy)|\Delta^\tau(y) < \infty \right\}. \tag{1}$$

It can be checked that this is a Banach space.

For $1 \leq p, q < +\infty$ and $\nu \in \mathbb{R}$, and $\nu > -1$ we denote by $A_\nu^{p,q}(T_\Omega)$ the mixed-norm weighted Bergman space consisting of analytic functions f in T_Ω such that

$$\|F\|_{A_\nu^{p,q}} = \left(\int_\Omega \left(\int_V |F(x+iy)|^p d\,x \right)^{q/p} \Delta^\nu(y) dy \right)^{1/q} < \infty,$$

this is a Banach space.

Replacing above A by L we will get as usual the corresponding larger space of all measurable functions in tube over symmetric cone with the same quazinorm (see [28], [31]).

It is known that the $A_\nu^{p,q}(T_\Omega)$ space is nontrivial if and only if $\nu > -1$, (see [29], [30]).

When $p = q$ we write (see [29])

$$A_\nu^{p,q}(T_\Omega) = A_\nu^p(T_\Omega).$$

This is the classical weighted Bergman space with usual modification when $p = \infty$.

The (weighted) Bergman projection P_ν is the orthogonal projection from the Hilbert space $L_\nu^2(T_\Omega)$ onto its closed subspace $A_\nu^2(T_\Omega)$ and it is given by the following integral formula (see [29])

$$P_\nu f(z) = C_\nu \int_{T_\Omega} B_\nu(z, w) f(w) dV_\nu(w), \tag{2}$$

where

$$B_\nu(z, w) = C_\nu \Delta^{-(\nu + \frac{n}{r})}((z - \overline{w})/i)$$

is the Bergman reproducing kernel for $A_\nu^2(T_\Omega)$, (see [28], [29]).

Here we used the notation $dV_\nu(w) = \Delta^{\nu - \frac{n}{r}}(v) du dv$. Below and here we use constantly the following notations $w = u + iv \in T_\Omega$ and also $z = x + iy \in T_\Omega$.

Hence for any analytic function from $A_\nu^2(T_\Omega)$ the following integral formula is valid (see also [29]).

$$f(z) = C_\nu \int_{T_\Omega} B_\nu(z, w) f(w) dV_\nu(w). \tag{3}$$

In this case sometimes below we say simply that the f function allows Bergman representation via Bergman kernel with ν index.

Note that these assertions have direct copies in simpler cases of analytic function spaces in unit disk, polydisk, unit ball, upperhalfspace C_+ and in spaces of harmonic functions in the unit ball or upperhalfspace of Euclidean space R^n These classical facts are well-known and can be found, for example, in [32], [27], [19] and in some items from references there.

Above and throughout the paper we write C (sometimes with indexes) to denote positive constants which might be different each time we see them (and even in a chain of inequalities), but are independent of the functions or variables being discussed.

The problem which we consider in this paper is classical (see, for example, [20–27]), we

wish to find sharp (or not) conditions on positive Borel measure μ in T_Ω so that

$$\int_{T_\Omega} |f(z)|^p d\mu(z) \le c\|f\|_Y, Y \subset H(T_\Omega),$$

where Y is a quasinormed subspace of $H(T_\Omega)$, $0 < p < \infty$.

We mention that in the case of the Hardy H^p space in the unit disk and for $H^p(B^k)$ in the ball such type result was obtained by D. LueckingL and P. Mercer with J. Cima in [14], [15] and L. Carleson (see [9], [10], [12]). The case of weighted Bergman spaces investigated in [16].

For Bergman space $Y = A_\alpha^p(T_\Omega)$, $0 < p < \infty$, $\alpha > -1$, (or Bergman type function spaces) this type of problem was considered before by various authors and solved for example in papers [23], [24]. For various other cases (spaces with more complicated norms or quasinorms) it is still open.

We mention a series of new sharp results of first author and authors of this paper (see [20] - [25]).

The plan of this paper is the following. We collect preliminaries and related assertions in our next section. In the third section we collect some known sharp results closely related with our work. The last section is devoted to some new embedding theorems. Note we use actively some machinery which was recently developed in [23], [24].

Note also some assertions of this paper were taken from our previous paper [23], [24], where some results were proved in less general situation namely in case of tubular domain in $\mathbb{C}^n$ and for one function case $m = 1$.

One of the intentions of this paper is to generalize them to multifunctional spaces tubular domains over symmetric cones.

Various related assertions (sharp embedding theorems in analytic function spaces in tubular domains over symmetric cones) can be seen in [23], [24].

The theory of analytic spaces on tubular domains over symmetric cones is well-developed by various authors during last decades (see [23], [24] and various references there). One of the goals of this paper among other things is to define for the first time in literature new mixed norm analytic spaces in tubular domains over symmetric cones and to establish some basic properties of these spaces. We believe that this new interesting objects can serve as a base for further generalizations and investigations in this active research area.

In the main part of paper we will turn to study of certain embedding theorems for some new mixed norm analytic classes in tubular domains over symmetric cones in $\mathbb{C}^n$. Proving estimates and embedding theorems in tubular domains we heavily use the technique which was developed recently in [23], [24]. In our embedding theorem and inequalities for analytic function spaces in tubular domains over symmetric cones the so-called Carleson-type measures constantly appear. We are ending with some historical remarks on this important topic now. Carleson measures were introduced by Carleson [9] in his solution of the corona problem in the unit disk of the complex plane, and, since then, have become an important tool in analysis, and an interesting object of study *per se.* Let A be a Banach space of holomorphic functions a domain $T_\Omega \subset \mathbb{C}^n$; given $p \ge 1$, a finite positive Borel measure μ on T_Ω is a *Carleson measure* of A (for p) if there is a continuous inclusion $A \to L^p(\mu)$, that is there exists a constant $C > 0$

such that

$$\forall f \in A \qquad \int\limits_{T_\Omega} |f|^p d\mu \leq C ||f||_A^p.$$

Carleson studied this property [9] taking as Banach space A the Hardy spaces in unit disk $H^p(\Delta)$, and proved that a finite positive Borel measure μ is a Carleson measure of $H^p(\Delta)$ for p if and only if there exists a constant $C > 0$ such that $\mu(S_{\theta_0,h}) \leq Ch$ for all sets

$$S_{\theta_0,h} = \left\{ re^{i\theta} \in \Delta : 1 - h \leq r < 1, \ \ |\theta - \theta_0| < h \right\}$$

(see also [10], [16]); in particular the set of Carleson measures of $H^p(\Delta)$ does not depend on p.

In 1975, W. Hastings [12] (see also V. Oleinik and B. Pavlov [16] and L. Oleinik [17]) proved a similar characterization for the Carleson measure of the Bergman spaces $A^p(\Delta)$, still expressed in terms of the sets $S_{\theta_0,h}$. Later J. Cima and W. Wogen [26] characterized Carleson measures for Bergman spaces in the unit ball $B^n \subset \mathbb{C}^k$, and J. Cima and P. Mercer [15] characterized Carleson measures of Bergman spaces in unit ball, showing in particular that the set of Carleson measures of $A^p(T_\Omega)$ is independent of $p \geq 1$.

2. Preliminaries on geometry of tubular domains over symmetric cones. In this section we will collect several very usefull assertions from [28], [29], [30], [31] mainly concerning so-called r-lattices that will be used rather often in all proofs of our sharp embedding theorems below.

Let $T_\Omega \subset \mathbb{C}^n$ be a bounded tubular domains over symmetric cones in $\mathbb{C}^n$. We shall use the following notations:

- $\delta : T_\Omega \to \mathbb{R}^+$ will denote the determinant function from the boundary, that is $\delta(z) = \Delta(Imz)$. Let $dv_t(z) = (\delta(z))^t dv(z)$, $t > -1$;
- v will be the Lebesgue measure on T_Ω;
- $H(T_\Omega)$ will denote the space of holomorphic on T_Ω, endowed with the topology of uniform convergence on compact subsets;
- $B : T_\Omega \times T_\Omega \to \mathbb{C}$ will be the Bergman kernel of T_Ω. Note that if B is kernel of type t, $t \in \mathbb{N}$, then B^s is kernel of type st, $s \in \mathbb{N}, t \in \mathbb{N}$. This follows directly from definition (see [23], [24], [28], [29], [30], [31]). Note $B = B_{2n/r}$ (see [23], [24], [28], [29], [30], [31]);
- given $r \in (0, \infty)$ and $z_0 \in T_\Omega$, we shall denote by $B_{T_\Omega}(z_0, r)$ the Bergman ball.

See, for example, [23], [24], [28], [29], [30], [31], for definitions, basic properties and applications to geometric function theory of the Bergman distance and [23], [24], [28], [29], [30], [31] for definitions and basic properties of the Bergman kernel. Let us now recall a number of vital results proved in T_Ω. The first two give information about the shape of Bergman balls:

Lemma 1. *(see [23], [24], [28], [29], [30], [31]) Let $T_\Omega \subset \mathbb{C}^n$ be a bounded tubular domains over symmetric cones, and $r \in (0, \infty)$. Then*

$$\nu(B_{T_\Omega}(\cdot, r)) \approx \delta^{2n/r},$$

where the constant depends on r.

Lemma 2. *(see [23], [24], [28], [29], [30], [31]) Let $T_\Omega \subset \mathbb{C}^n$ be a bounded tubular domains*

over symmetric cones. Then there is $C > 0$ such that

$$\frac{C}{1-r}\delta(z_0) \leq \delta(z) \leq \frac{1-r}{C}\delta(z_0)$$

for all $r \in (0, \infty)$, $z_0 \in T_\Omega$ and $z \in B_{T_\Omega}(z_0, r)$.

Definition 1. *Let $T_\Omega \subset \mathbb{C}^n$ be a tubular domains over symmetric cones, and $r > 0$. An r-lattice in T_Ω is a sequence $\{a_k\} \subset T_\Omega$ such that $T_\Omega = \bigcup_k B_{T_\Omega}(a_k, r)$ and there exists $m > 0$ such that any point in T_Ω belongs to at most m balls of the form $B_{T_\Omega}(a_k, R)$, where $R = \frac{1}{2}(1 + r)$.*

Note by Lemma 2, $v_\alpha(B_{T_\Omega}(a_k, R)) = \int\limits_{B_{T_\Omega}(a_k, R)} \delta^\alpha(z)dv(z) = (\delta^\alpha(a_k))v(B_{T_\Omega}(a_k, R))$, $\alpha > -1$.

The existence of r-lattice intubular domains over symmetric cones is ensured by the following.

Lemma 3. *(see [23], [24], [28], [29], [30], [31], [38], [39]) Let $T_\Omega \subset \mathbb{C}^n$ be a tubular domains over symmetric cones. Then for every $r \in (0, \infty)$ there exists an r-lattice in T_Ω, that is there exists $m \in \mathbb{N}$ and a sequence $\{a_k\} \subset T_\Omega$ of points such that $T_\Omega = \bigcup\limits_{k=0}^{\infty} B_{T_\Omega}(a_k, r)$ and no point of T_Ω belongs to more than m of the balls $B_{T_\Omega}(a_k, R)$, where $R = \frac{1}{2}(1 + r)$.*

We will call r-lattice sometimes the family $B_{T_\Omega}(a_k, r)$. Dealing with B Bergman kernel we always assume $|B(z, a_k)| \asymp |B(a_k, a_k)|$ for any $z \in B_{T_\Omega}(a_k, r)$, $r \in (0, \infty)$ (see [23], [24], [28], [29], [30], [31]). Let $m = (2n/r)l, l \in \mathbb{N}$. Then $|B_m(z, a_k)| \asymp |B_m(a_k, a_k)|$, $z \in B_{T_\Omega}(a_k, r)$, $r \in (0, \infty)$. This fact is crucial for embedding theorems in tubular domains over symmetric cones(see also [21], [25]).

We shall use a submean estimate for non-negative plurisubharmonic functions on Bergman balls:

Lemma 4. *(see [23], [24], [28], [29], [30], [31], [38], [39]) Let $T_\Omega \subset \mathbb{C}^n$ be a tubular domains over symmetric cones. Given $r \in (0, \infty)$, set $R = \frac{1}{2}(1 + r) \in (0, \infty)$. Then there exists a $C_r > 0$ depending on r such that*

$$\forall z_0 \in T_\Omega, \quad \forall z \in B_{T_\Omega}(z_0, r), \quad \chi(z) \leq \frac{C_r}{v(B_{T_\Omega}(z_0, r))} \int\limits_{B_{T_\Omega}(z_0, R)} \chi dv$$

for every nonnegative plurisubharmonic function $\chi : T_\Omega \to \mathbb{R}^+$.

We will use this lemma for $\chi = |f(z)|^q$, $f \in H(T_\Omega)$, $q \in (0, +\infty)$.

Lemma 5. *(see [38], [39])*

1) Let $\lambda > \frac{n}{r} - 1$ be fixed. Then $\Delta(y + y') \geq \Delta(y); \forall y, y' \in \Omega, |\Delta^{-\lambda}(\frac{x+iy}{i})| \leq (\Delta(y)^{-\lambda})$; $\forall x \in \mathbb{R}^n; y \in \Omega$.

2) Let α, β are real, then

$$I_{\alpha,\beta}(t) = \int_\Omega (\Delta^\alpha(y + t))(\Delta^\beta(y))dy < \infty,$$

if $\beta > -1, \alpha + \beta < (-\frac{2n}{r} + 1)$, and

$$I_{\alpha,\beta}(t) = (c_{\alpha,\beta})\Delta^{\alpha+\beta+\frac{n}{r}}(t).$$

Moreover

$$I_\alpha(y) = \int_{\mathbb{R}^n} |(\Delta^{-\alpha}(\frac{x+iy}{i})|dx < \infty,$$

if $\alpha > \frac{2n}{r} - 1$; and

$$I_\alpha(y) = c_\alpha(\Delta^{-\alpha+\frac{n}{r}}(y)),$$

where $y \in \Omega$.

Lemma 6. *For any analytic function from $A_\alpha^2(T_\Omega)$ we following integral formula is valid*

$$f(z) = \tilde{c}_\alpha \int_{T_\Omega} B_\alpha(z,w)(f(w))dv_\alpha(w), z \in T_\Omega. \qquad (*)$$

Let $1 \le p < \infty, 1 \le q < \infty, \frac{n}{r} \le p_1, \frac{1}{p_1} + \frac{1}{p} = 1, \frac{n}{r} < \gamma$.

Let $f \in A_\gamma^{p,q}$, then $(*)$ with $\alpha > \frac{n}{r} - 1$ is valid (Bergman representation formula with α index is valid).

We now collect a few facts on the (possibly weighted) L^p-norms of the Bergman kernel and the normalized Bergman kernel. The first result is classical (see, for example, [28], [29], [31]).

The first result is the main result of this section, and contains the weighted L^p-estimates we shall need (Forelly-Rudin estimates):

Proposition 1. *Let $T_\Omega \subset \mathbb{C}^n$ be a tubular domains over symmetric cones, and let $z_0 \in T_\Omega$ and $1 \le p < \infty$. Then*

$$\int_{T_\Omega} |B(\zeta, z_0)|^p \delta^\beta(\zeta) d\nu(\zeta) \preccurlyeq \delta^{\beta - (2n/r)(p-1)}(z_0), -1 < \beta < (2n/r)(p-1).$$

The same result is valid for weighted Bergman kernel (see [23]).

3. Preliminary theorems. In this section we first review some known sharp embedding theorem in tubular domains over symmetric cones with smooth boundary.

Let first $\beta, \theta \in R, p > 0$. A (analytic) Carleson measure of $A^p(T_\Omega, \beta) = A_\beta^p$ is a finite positive Borel measure on T_Ω such that there is a positive continuous inclusion $A^p(T_\Omega, \beta) \subset L^p(\mu)$ that is there is a constant $c > 0$ such that $\int_{T_\Omega} |f|^p d\mu \le c\|f\|_{A_\beta^p}^p, \forall f \in A^p(T_\Omega, \beta), \theta$ Carleson measure is a finite positive Borel measure on T_Ω such that $\mu(B_{T_\Omega}(\cdot, r)) \le cv(B_{T_\Omega}(\cdot, r))^\theta$ for all $r \in (0,1)$ where the constant c might depend on r. If $\theta = 1$ we have usual Carleson measure. ($\beta = 0$ case).

We first list two known sharp embeddings in this direction (see [23], [24] and various references there).

Theorem A. Let μ be a positive Borel measure on T_Ω, $f \in H(T_\Omega)$. Let $1 \le p < \infty$. We have $\int_{T_\Omega} |f|^p d\mu \le c\|f\|_{A^p}^p$ iff $\mu(B_{T_\Omega}(a_k, r)) \le v(B_{T_\Omega}(a_k, r)), r > 0$ or if $\mu(B_{T_\Omega}(\cdot, r)) \le cv(B_{T_\Omega}(\cdot, r))$ or if $\mu(B_{T_\Omega}(a_k, r)) \le c(\delta^{2n/r}(a_k))$ for certain sequence $\{a_k\}$ which is $r-$lattice for T_Ω.

This vital theorem was extended recently (see [23], [24] and various references there).

Theorem B. Let μ be a positive Borel measure on T_Ω, $f \in H(T_\Omega)$. Let $\theta \geq 1$, $1 \leq p < \infty$. Then the following assertions are equivalent

1. $\int\limits_{T_\Omega} |f(z)|^p d\mu(z) \leq c \int\limits_{T_\Omega} |f(z)|^p \delta^{(2n/r)(\theta-1)}(z) dv(z)$,

2. μ is $\theta-$Carleson measure,

3. for every $r \in (0,1)$ and every $r-$lattice $\{a_k\}$ in T_Ω one has $\mu(B_{T_\Omega}(a_k, r)) \leq [v(B_{T_\Omega}(a_k, r))]^\theta$, $r > 0$.

4. there exists $r_0 \in (0,1)$ so that for every r_0-lattice $\{a_k\}$ in T_Ω, $\mu(B_{T_\Omega}(a_k, r_0)) \leq c[v(B_{T_\Omega}(a_k, r_0))]^\theta$.

The following theorem is another sharp embedding theorem for mixed norm spaces.

In the unit ball case Theorem C can be seen in [20]. The proof of tubular domain case is the same as in [20], [21] for the unit ball case and pseudoconvex domains based on some embeddings from [36].

Theorem C. (see [22]) Let μ be positive Borel measure on T_Ω, $f \in H(T_\Omega)$. Let $\{a_k\}$ be $r-$lattice. Assume $q < p$ or $q = p$, $r \leq p$. Then

$$\left(\int\limits_{T_\Omega} |f(z)|^p d\mu(z) \right)^{1/p} \leq c_0 \|f\|_{A_v^{q,\tau}}$$

if $\mu(B_{T_\Omega}(a, r)) \leq c_1 \delta^\tau(a)$, $a \in T_\Omega$ or if $\mu(B_{T_\Omega}(a_k, r)) \leq c_2 \delta^\tau(a_k)$ for $k = 1, 2, \ldots$ and for some constants c_1, c_2, and for some $\tau = \tau(p, q, v)$.

We also refer the reader to [1–4], [10], [15–17], [20–27], where new sharp interesting results (embedding theorems) in this direction were recently provided.

4. Main results. We show first our two main theorems in this section. Namely, we show new two sharp embedding theorems and our arguments are mostly standard and sketchy. Results from our point are interesting enough and enlarge the list of previously known sharp results in this direction.

We define new analytic general multifunctional Herz (for $p = q$ Bergman) spaces in tubular domains over symmetric cones with smooth boundary as spaces of $(f_k)_{k=1}^m$ functions analytic in T_Ω, so that

$$\|(f_1, \ldots, f_m)\|_{A(p,q,m,d\mu)}^q = \sum_{k=1}^\infty \left(\int_{B(a_k,r)} \prod_{i=1}^m |f_i(z)|^p d\mu(z) \right)^{q/p} < \infty, \ 0 < p, q < \infty,$$

μ is a positive Borel measure on T_Ω.

Note for $m = 1$, $q = p$ in the unit ball, polydisk, tubular domains over symmetric cones these spaces are well-known (see [23], [24] and references there). For $m = 1$ these are known Herz spaces.

For $d\mu = \delta^s(z)dv(z) = dv_s(z)$ we write $A(p,q,m,s)$ and if $m = 1$ we write $A(p,q,d\mu)$ or $A(p,q,s)$ if $d\mu = dv_s$. Also, we put, for convenience, below $\Delta_k = B(a_k, r)$ and we put $\Delta_k^* = B(a_k, R)$, $R = \dfrac{1+r}{2}$, $k = 1, 2, \ldots$

We put some additional condition below on Bergman kernel now which probably can be

removed.

Even this additional condition based on some recent results of B. Sehba [37] can be dropped. In this paper we, for $\alpha q > \beta + 2n/r$, always assume (this is true in the ball, see [27] for $\beta > 0$)

$$\int_{B(\widetilde{z},r)} |B_\alpha(z,w)|^q dv_\beta(w) \leq \widetilde{C}|B_{\alpha q - \beta - 2n/r}(z,\widetilde{z})|,$$

for all $\beta > -1$, $q > 0$, $\alpha > 0$, namely in Theorems 3 and 5, $z, \widetilde{z} \in T_\Omega$.

The proof of the necessity of condition on measure in all theorems uses standard arguments with standard test function (see, for example, [23], [24]). We put for some $\tau > 0$

$$f_i(z) = f_w(z) = B_\tau(w,z), \ i = 1,\ldots,m, \ z,w,a_k \in T_\Omega$$

and reduce the problem to onefunctional case noting that

$$\|(f_1,\ldots,f_m)\|_{A(p,q,m,d\mu)} = \|(f_w,\ldots,f_w)\|_{A(p,q,m,d\mu)} \geq$$

$$\geq c \left(\int_{\Delta_k} |f_w(z)|^{mp} d\mu(z) \right)^{\frac{1}{p}} \geq c \left(\mu(\Delta_k) \right)^{v_1} \left(\delta(a_k) \right)^v,$$

for some $v, v_1 > 0$ and for some fixed $\{a_k\}$ r-lattice or

$$\int_{T_\Omega} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \geq c \int_{\Delta_k} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \geq \widetilde{c} \left(\mu(\Delta_k) \right)^{\widetilde{v_1}} \left(\delta(a_k) \right)^{\widetilde{v}},$$

for some $\widetilde{v_1}, \widetilde{v} > 0$ (depending on parameters involved). On the other hand, to estimate norms of test function in mixed norm $A_\alpha^{p,q}$ or related A_α^p Bergman spaces the condition on kernel we put (see below) and then Forrelly-Rudin type estimates must be used. We will partial omit these standard details.

Theorem 1. *Let $0 < p, q < \infty$, $0 < s \leq q$, $\beta_j > -1$ for $j = 1,\ldots,m$ and let μ be a positive Borel measure on T_Ω. Then the following conditions are equivalent:*

1. If $f_i \in A_{\beta_i}^s$, $i = 1,\ldots,m$, then

$$\|(f_1,\ldots,f_m)\|_{A(p,q,m,d\mu)} \leq C \prod_{i=1}^m \|f_i\|_{A_{\beta_i}^s}. \tag{4}$$

2. The measure μ satisfies a Carleson type condition:

$$\mu(\Delta_k) \leq C\delta(a_k)^{\sum_{i=1}^m \frac{p(2n/r + \beta_j)}{s}}, \ k \geq 1,$$

for any $\{a_k\}$ r-lattice in T_Ω.

The corresponding more general result for mixed norm spaces is the following theorem.

Theorem 2. *Let $0 < p, q < \infty$, $0 < s \leq q$, $\beta_j > -1$ for $j = 1,\ldots,m$, $0 < t_i \leq s$ for $i = 1,\ldots,m$, $\sum_{i=1}^n 1/t_i = m/s, m \in \mathbb{N}$, and let μ be a positive Borel measure on T_Ω.*

Then the following conditions are equivalent:

1. *If $f_i \in A^{s,t_i}_{\beta_i}$, $i = 1, \ldots, m$, then*

$$\|(f_1, \ldots, f_m)\|_{A(p,q,m,d\mu)} \leq C \prod_{i=1}^{m} \|f_i\|_{A^{s,t_i}_{\beta_i}}. \qquad (5)$$

2. *The measure μ satisfies a Carleson type condition:*

$$\mu(\Delta_k) \leq C\delta(a_k)^{\sum_{i=1}^{m} \frac{p(2n/r+\beta_j)}{s}}, \ k \geq 1,$$

for any $\{a_k\}$ r-lattice in T_Ω.

The following is another new sharp theorem for multifunctional analytic spaces in tubular domains over symmetric cones.

Theorem 3. *Let $0 < p, q < \infty$, $0 < \sigma_i \leq q$, $-1 < \alpha_i < \infty$ for $i = 1, \ldots, m$. Let μ be a positive Borel measure on T_Ω. Then the following two conditions are equivalent:*

1.

$$\|(f_1, \ldots, f_m)\|^q_{A(p,q,m,d\mu)} \leq C \prod_{i=1}^{m} \int_{T_\Omega} \left(\int_{B(w,r)} |f_i(z)|^{\sigma_i} \delta^{\alpha_i}(z) dv(z) \right)^{q/\sigma_i} dw. \qquad (6)$$

2. *The measure μ satisfies a Carleson type condition:*

$$\mu(\Delta_k) \leq C\delta(a_k)^{m(2n/r)\frac{p}{q}+\sum_{i=1}^{m} \frac{p(2n/r+\alpha_i)}{\sigma_i}}, \ k \geq 1,$$

for any $\{a_k\}$ r-lattice in T_Ω.

In our Theorems 4 and 5 below we consider another two new scales of Herz-type spaces (multifunctional) defined with the help of the following expressions:

$$\prod_{i=1}^{m} \left(\sum_{i=1}^{\infty} \left(\int_{\Delta_k^*} |f_i(z)|^{p_i} \delta^{\alpha}(z) dv(z) \right)^{q_i} \right)^{\frac{1}{q_i}},$$

$$\prod_{i=1}^{m} \int_{T_\Omega} \left(\int_{B(w,r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}},$$

and prove some sharp assertions also for them. Proofs here will be sketchy since ideas we already provided will be repeated partially by us below.

Theorem 4. *Let μ be a positive Borel measure on T_Ω. Let $0 < p_i, q_i < \infty$, $i = 1, \ldots, m$, let $\sum_{i=1}^{m} \frac{1}{q_i} = 1$, $\alpha > -1$. Then the following conditions are equivalent:*

1) *If $f_i \in H(T_\Omega)$, $i = 1 \ldots m$ then*

$$\int_{T_\Omega} \prod_{i=1}^{m} |f_i(z)|^{p_i} d\mu(z) \leq c \prod_{i=1}^{m} \left(\sum_{i=1}^{\infty} \left(\int_{\Delta_k^*} |f_i(z)|^{p_i} \delta^{\alpha}(z) dv(z) \right)^{q_i} \right)^{\frac{1}{q_i}}.$$

2) *The measure μ satisfies a Carleson type condition $\mu(\triangle_k) \leq C|\triangle_k|^{m\left(1+\frac{\alpha}{2n/r}\right)}$ for $k \geq 1$.*

Remark 1. Our Theorem 4 for bounded pseudoconvex domains can be seen in [25]. The

same proof can be given in bounded symmetric domains, based on properties of r-lattices on these domains. Note also the additional condition on Bergman kernel used in the proof of Theorems 3,5 can be removed based on some results of Sehba (see [31], [36]).

The following additional condition on analytic $(f_i)_{i=1}^m$ functions (which probably can be removed) is needed for our last theorem with condition on kernel (see before Theorem 1). We assume that for any Bergman ball $B(w, r)$

$$(\delta(a_k))^{2n/r} \left(\iint_{B^*(a_k,r)} |f_i(z)|^{\sigma_i} \right)^{\frac{p_i}{\sigma_i}} dv_{\alpha_i}(z) \leq \widetilde{c} \int_{B(a_k,r)} \left(\int_{B(w,r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w),$$

where $\{a_k\}$ is any r-lattice, $0 < p_i, \sigma_i < \infty, \alpha_j > -1, j = 1, \ldots, m, dv_\alpha = \delta^\alpha(z)dv(z)$.

Theorem 5. *Let $0 < p_i, \sigma_i < \infty, -1 < \alpha_i < \infty, i = 1, \ldots, m$. Let μ be a positive Borel measure on T_Ω. Then the following conditions are equivalent:*

1) If $f_i \in H(T_\Omega), i = 1, \ldots, m$ then

$$\int_{T_\Omega} \prod_{i=1}^m |f_i(z)|^{p_i} d\mu(z) \leq c \prod_{i=1}^m \int_{T_\Omega} \left(\int_{B(w,r)} |f_i(z)|^{\sigma_i} dv_{\alpha_i}(z) \right)^{\frac{p_i}{\sigma_i}} dv(w).$$

2) The positive Borel measure μ satisfies the following Carleson type condition: $\mu(\triangle_k) \leq c_1 \delta(a_k)^\tau$, where $\tau = m(2n/r) + \sum_{i=1}^m \dfrac{p_i(2n/r + \alpha_i)}{\sigma_i}, k \geq 1$ for some constants c and c_1, and for any $\{a_k\}$ r-lattice in T_Ω.

Remark 2. Very similar results with similar proofscan be shown for similar type spaces in bounded strongly pseudoconvex domains (see [18], [21–26], for one functional case for similar results), based on known properties of r-lattices on these domains (see [1], [2]).

Remark 3. Very similar sharp results with similar proofs can be provided (under some condition on kernel) in bounded symmetric domains in $\mathbb{C}^n$ based on known properties of r-lattices on these domains (see [6]).

Remark 4. Note in [3], [4] complete analogues of our embedding theorems can be seen in harmonic function spaces.

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ABSTRACT

We introduce new multifunctional mixed norm analytic Herz-type spaces in tubular domains over symmetric cones and provide new sharp embedding theorems for them. Some results are new even in case of onefunctional holomorphic spaces. Some new related sharp results for new multifunctional Bergman-type spaces will be also provided under one condition on Bergman kernel.

Keywords: Bergman spaces, Herz spaces, tubular domains over symmetric cones, embedding theorems, analytic functions.

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АННОТАЦИЯ

В работе вводится новый многофункциональный аналитический тип Герца со смешанной нормой пространства в трубчатых областях над симметричными конусами и приводятся для них новые точные теоремы вложения. Некоторые результаты являются новыми даже в случае однофункциональных голоморфных пространств. Некоторые новые связанные точные результаты для новых многофункциональных пространств типа Бергмана будут также предоставлены при одном условии на ядро Бергмана.

Ключевые слова: пространства Бергмана, пространства Герца, трубчатые области над симметричными конусами, теоремы вложения, аналитические функции.

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