## MATHEMATICS

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# On traces of new BMOA type spaces in tubular domains over symmetric cones

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В данной мы приводим новые точные теоремы о следах аналитических классов типа BMOA в трубчатых областях над симметрическими конусами при одном дополнительном условии на так называемое ядро Бергмана в этих областях. Существенную роль при этом играют так называемые r -решетки для трубчатых областей, полученные ранее в работах зарубежных авторов. Теоремы данной работы являются полными аналогами наших недавних точных результатов о следах класов BMOA,полученные ранее авторами в полишаре и в ограниченных строго псевдовыпуклым областях.

**Ключевые слова:** *Трубчатые области,классы типа BMOA,аналитические функции.,следы классов и функций*

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## Introduction

Let $\Omega$ be a tubular domain over symmetric cones with smooth boundary in $\mathbb{C}^n$. Let $r$ - be the rank of the cone of our tubular domain (see [10]–[12]). By $\delta$ we denote as usual the well-known so-called determinant function $\Delta(z)$ in tubular domain over symmetric cone (see for this function and it is properties [4] and references there). Let further $\mathcal{H}(\Omega)$ denote the space of all holomorphic functions on $\Omega$ endowed as usual with the topology of uniform convergence on compact subsets. $V(z)$ will be the Lebesgue measure on the domain $\Omega$. $B(z,w)$ or $B_{2n/r}(z,w)$ will be the Bergman kernel of order $2n/r$ of $\Omega$ (see [4]). For positive $\alpha$ the Bergman kernel of order $\alpha$ will be denoted by $B_\alpha(z,w)$ (see [4]). For given $\widetilde{r} \in (0,1)$ and $z_0 \in \Omega$, we shall denote by $B_\Omega(z_0,\widetilde{r})$ the Bergman ball of center $z_0$ and the radius $\widetilde{r}$. We refer

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the reader [4] for definitions, basic properties and applications to geometric function theory of the Bergman distance and for definition and basic properties of the Bergman kernel. We will need several results for our proofs about Bergman balls for which we refer the reader to [4],[10]–[12]. In particular we constantly use among other things the fact that $V(B_\Omega(z,\widetilde{r}))$ is equivalent to $\delta^{2n/r}(z)$, $\widetilde{r}\in(0,1)$. We denote $m$ products of tubular domains by $\Omega^m$, the space of all analytic functions on this new product domain which are analytic by definition in each variable separately will be denoted by $\mathcal{H}(\Omega^m)$.

Let now $\widetilde{m}=2,3,\dots$ be a natural number, $M\subset\mathbb{C}^n$ and $B\subset\mathbb{C}^{\widetilde{m}n}$, $\mathbb{C}^{\widetilde{m}n}=\mathbb{C}^n\times\dots\times\mathbb{C}^n$, be a subset. Let $X(M)$ be a class of functions on $M$, $Y(B)$ the same. We say $TraceY(M^m)=X(M)$, $B=M^m$, $M^m=M\times\dots\times M$, if for any $f\in Y(M^m)$, $f(w,\dots,w)\in X(M)$, $w\in M$, and for any $g\in X(M)$, there exist a function $f\in Y(B)$ such that $f(w,\dots,w)=g(w)$, $w\in M$. Traces of various functional spaces in $\mathbb{R}^n$ were described in [9]. In polydisk this problem is also known as a problem of diagonal map (see [6] and references there).

Trace theorems even for $n=1$ (case of polydisk) have numerous applications in the theory of holomorphic functions (see, for example, [6]). The core of our proof is a special composition estimate an analogue of Andersson-Carlsson composition estimate (see [3]).

We say $A\approx B$ ($A$ is equivalent to $B$) if there are positive constants $C_1$ and $C_2$ so that $C_1A\le B\le C_2A$.

The authors proved many such type results on traces in recent years.

## Preliminaries

In this section we collect various known preliminaries on the topic of this short note. Let

$$A^\infty_\tau(\Omega)=\left\{F\in\mathcal{H}(\Omega):\|F\|_{A^\infty_\tau}=\sup_{z\in\Omega}|F(z)|\delta^\tau(z)<\infty\right\}, \tag{1}$$

(see [4], [10]–[12] and references there). It can be checked that this is a Banach space.

For $1<p<+\infty$ and $\nu\in\mathbb{R}$ and $\nu>-1$ we denote by $A^p_\nu(\Omega)$ the weighted Bergman space consisting of analytic functions $f$ in $\Omega$ such that

$$\|F\|^p_{A^p_\nu}=\left(\int_\Omega|F(z)|^p dV_\nu(z)\right)<\infty.$$

Here we used the notation $dV_\nu(w)=\delta^\nu dV(w)$. This space is a Banach space.

To define two related Bergman-type spaces $A^p_\nu(\Omega^m)$ and $A^\infty_\tau(\Omega^m)$ ($\nu$ and $\tau$ can be also vectors) in products of $m$ copies of tubular domains $\Omega^m$ (here all domains are the same) we follow standard procedure which is well-known in case of unit disk and unit ball (see [14]). Namely we consider analytic functions $F=F(z_1,\dots z_m)$, where each coordinate varies in a tubular domain (spaces on product domains). It can be shown that these are also Banach spaces.

The (weighted) Bergman projection $P_\nu$ is the orthogonal projection from the Hilbert space $L^2_\nu(\Omega)$ onto its closed subspace $A^2_\nu(\Omega)$ and it is given by the following integral formula (see, for example, [4], [10]–[12]).

$$P_\nu f(z) = C_\nu \int_\Omega B_{\nu+2n/r}(z,w)f(w)dV_\nu(w), \tag{2}$$

where $B_{\nu+2n/r}(z,w)$ is the Bergman reproducing kernel for $A^2_\nu(\Omega)$ (see [4], [10]–[12]), where $dV_\nu(w) = \delta^\nu dV(w)$, $\nu > -1$ and where $C_\nu$ is a constant of the Bergman representation formula. For any analytic function in $A^2_\nu(\Omega)$ the following integral formula is valid (see also [4])

$$f(z) = C_\nu \int_\Omega B_{\nu+2n/r}(z,w)f(w)dV_\nu(w),\ z \in \Omega. \tag{3}$$

In this case we say sometimes below simply that the $f$ function allows the Bergman representation via Bergman kernel with $\nu$ index. The existence of suitable coverings ($r$-lattices) of a domain $\Omega$ based on Bergman balls is crucial for results of this note. Note these assertions (assertions on the Bergman kernel and assertions on the $r$-lattices) have direct analogues in simpler cases of analytic function spaces in unit disk, polydisk, unit ball, upper half-space $\mathbb{C}_+$ and in spaces of harmonic functions in the unit ball or upper half-space of the Euclidean space $\mathbb{R}^n$. These classical facts are well-known and can be found, for example, in [6], [8], [16] and in some items from references there.

Let now $\Omega$ be a tubular domain over symmetric cones.

We will need for our proofs the following important fact on integral representations (see [8], [10], [13]). For all $1 \le p < \infty$, $\nu > -1$ and for all $f$ functions from $A^p_\nu$ the Bergman representation formula with $\alpha + 2n/r$ index or with the Bergman kernel $B_{\alpha+2n/r}(z,w)$ is valid, for all $\alpha$, $\alpha > \alpha_0$, for certain fixed $\alpha_0$, (see [4], [10]–[12]). Let $\alpha > -1$ then for all $\nu > \nu_0$ for certain fixed $\nu_0$ and all $f$ functions $f \in A^\infty_\alpha$ the integral representations of Bergman with Bergman kernel $B_{\nu+2n/r}(z,w)$ (with $\nu + 2n/r$ index) is valid. We note also that (see [4], [10]–[12])

$$|f(z)|\delta^{\frac{2n/r+\nu}{p}}(z) \le c_{p,\nu}\|f\|_{A^p_\nu}, \quad z \in \Omega. \tag{4}$$

All the mentioned results together with properties of the Whitney decomposition ($r$-lattice) of tubular domains based on Bergman balls in $\mathbb{C}^n$ are vital for this paper. Proofs of our theorems on trace in BMOA in the ball are heavily based on properties of $r$-lattice $\{a_k\}$ in a Bergman metric (see [10]–[13]). Our proofs below are also based on same ideas as in the unit ball (see [14]) and is based also on same properties of lattices but on tubular domains (see [4], [10]–[12]).

**Definition 1.** *( [4], [10]–[12]) Let $\Omega \subset \mathbb{C}^n$ be a tubular domain and $r > 0$. An $r$-lattice in $\Omega$ is a sequence $\{a_k\} \subset \Omega$ such that $\Omega = \bigcup_k B_\Omega(a_k, \widetilde{r})$ and there exists $m > 0$, such that any point in $\Omega$ belong to at most $m$ balls of the form $B_\Omega(a_k, R)$, where $R = \frac{1}{2}(1+\widetilde{r})$.*

The following lemmas are vital for proofs of main results of this paper.

**Lemma 1.** *( [4], [10]–[12]) Let $\Omega \subset \mathbb{C}^n$ be a tubular domain over symmetric cone. Then for every $\widetilde{r} \in (0,1)$ there exists an $r$-lattice in $\Omega$, that is there exists $m \in \mathbb{N}$ and a sequence $\{a_k\} \subset \Omega$ of points such that $\Omega = \cup_{k=0}^\infty B_\Omega(a_k, \widetilde{r})$ and no point of $\Omega$ belongs to more than $m$ of the balls $B_\Omega(a_k, R)$.*

**Lemma 2.** *( [4], [10]–[12]) Let $\Omega \subset \mathbb{C}^n$ be a tubular domain over symmetric cone. Given*

$\widetilde{r} \in (0,1)$. *Then there exists a* $k_r > 0$ *depending on* $r$ *such that for all* $z_0 \in \Omega$ *and for all* $z \in B_\Omega(z_0, \widetilde{r})$

1. $f(z) \leq \frac{k_r}{V(B_\Omega(z_0,\widetilde{r}))} \int_{B_\Omega(z_0,\widetilde{r})} (f(w))dV(w)$,
2. $V(B_\Omega(\cdot, \widetilde{r})) \approx \delta^{2n/r}$,

*for every nonnegative holomorphic function* $f : \Omega \to \mathbb{R}^+$.

We now provide known Forelli - Rudin estimates in the tube (see, for example, [10]–[12]).

**Lemma 3.** *( [4], [10]–[12]) Let* $\Omega \subset \mathbb{C}^n$ *be a tubular domain over symmetric cone and let* $z_0 \in \Omega$ *and* $1 \leq p < \infty$. *Then we have the following estimate for Bergman type kernel* $B_{2n/r}(z, z_0)$

$$\int_\Omega |B_{2n/r}(\xi, z_0)|^p \cdot \delta(\xi)^\beta dV(\xi) \lesssim \tag{5}$$

$$\lesssim \ \delta(z_0)^{\beta-(2n/r)(p-1)}, \ \ -1 < \beta < (2n/r)(p-1), \ z_0 \in \Omega.$$

Estimate (5) is valid for all $B_t$ kernels (see [4], [10]–[12]).

We suppose below that the following additional condition on the Bergman kernel is valid. We note this condition probably can be dropped. Note also this condition is valid in $R^{n+1}$ and in the unit ball and even in bounded strongly pseudoconvex domains with smooth boundary in $C^n$ (see for this issues [14], [15] and also various references there).

We will assume that the following composition estimates in tube is valid. Note it is valid in the unit ball and bounded strongly pseudoconvex domains. Assume that $\alpha > 0$, $\beta > 0$, $\widetilde{r} > -1$, $\alpha - \widetilde{r} < 2n/r$, $\beta - \widetilde{r} < 2n/r$ and $\alpha + \beta - \widetilde{r} > 2n/r$. Then

$$\int_\Omega |B_\alpha(\zeta, z)||B_\beta(\zeta, w)|\delta^{\widetilde{r}}(\zeta)dV(\zeta) \lesssim |B_{\alpha+\beta-\widetilde{r}-2n/r}(z, w)|, \ z, w \in \Omega. \tag{6}$$

If instead $\beta - \widetilde{r} > 2n/r$ we have

$$\int_\Omega |B_\alpha(\zeta, z)||B_\beta(\zeta, w)|\delta^{\widetilde{r}}(\zeta)dV(\zeta) \lesssim \frac{|B_\alpha(z, w)|}{\delta^{\beta-\widetilde{r}-2n/r}(w)}, \ z, w \in \Omega. \tag{7}$$

For our BMOA type spaces (see definition below) in a domain (not in product domains) the Bergman representation formula with large enough index is valid. Note, also, in our Theorem 2, we assume that $|B_t(z, a_k)|$ is equivalent to $|B_t(w, a_k)|$ for any Bergman kernel of $t$ type for any $w$ and $z$ in $B_\Omega(a_m, \widetilde{r})$ and any $a_k$, $k \in \mathbb{N}$, where $m$ any natural number (this is an additional condition on the Bergman kernel which is valid (see also discussion below) in the unit ball, see [17]. This condition in the unit ball can be seen in [17] and also plays a key role for the proof of our theorems in the unit ball case (see [14]). This additional condition on the Bergman kernel will be used only in the proof of our main theorems for $p \leq 1$ case. This condition concerning Bergman kernel is valid in milder form (see [4], [10]–[12] and references there). The condition on integral representation is valid with some restriction on indexes for our BMOA type spaces in tubular domains with smooth boundary. This follows from the fact that these spaces with some restriction on indexes are embedded in Bergman space $A^1_\nu$ with large enough $\nu$ and for this spaces this integral representation is valid. The short proof of this last fact follows by similar arguments as in the case of the unit disk (see [6], [16], [17]).

Namely, the short proof of this follows from crucial estimate from below of Bergman kernel on Bergman ball in tubular domains (see, for example, [10] and references there and [10]–[13]).

## Main results

In this section we formulate our main trace theorems. In case of unit ball they were proved in [14]. We recently extended some of our trace theorems from unit polyball to pseudconvex domains. In this paper we provide such extension for our another trace result in unit polyball, namely for so called analytic BMOA type spaces. First we introduce a BMOA type space in tubular domains then formulate our trace theorems. BMOA type spaces in tubular and other domains gathered a lot of attention recently (see [3], [8], [14], [15] and various references there).

Let $M^p_{r_1,\dots,r_m,\tau,s_1,\dots,s_m}(\Omega^m) =$

$$= \left\{ f \in \mathcal{H}(\Omega^m) : \sup_{w\in\Omega} \delta^{m\tau}(w) \int_\Omega \cdots \int_\Omega |f(z_1,\dots,z_m)|^p \prod_{j=1}^m |B_{S_j}(z_j,w)| \prod_{j=1}^m \delta^{s_j}(z_j) dV(z_j) < \infty \right\},$$

$S_j = \tau + r_j, j = 1,\dots,m$, where $\tau > 0,\ s_j > -1,\ r_j \geq 0,\ j = 1,\dots,m,\ p \in (0,\infty)$.

These are Banach spaces for all $p \geq 1$ and complete metric spaces for other values of $p$.

The following theorems for unit ball case can be seen in [14]. The proofs hinges on similar arguments as in unit ball from one hand and on recent advances in analytic function theory in tubular domains (see [10]–[12]).

**Theorem 1.** *Let $p > 1$, $\tau \in (0,\infty)$, $r_j \in \mathbb{N},\ s_j > -1,\ j = 1,\dots,m$. If $t = (m-1)(2n/r) + \sum_{j=1}^m s_j$, then for $r = \sum_{j=1}^m r_j$ we have*

$$Trace(M^p_{r_1,\dots,r_m,\tau,s_1,\dots,s_m}(\Omega^m)) = M^p_{r,t,\tau m}(\Omega)$$

*for all $n,\ n/r > n_0$, where $n_0 = n_0(p,\tau,r_1,\dots,r_m,m)$.*

**Theorem 2.** *Let $p \leq 1$, $\tau \in (0,\infty)$, $r_j \in \mathbb{N},\ s_j > -1,\ j = 1,\dots,m$. If $r_j/p \in \mathbb{N}$, $j = 1,\dots,m$ and $t = (m-1)(2n/r) + \sum_{j=1}^m s_j$, then for $r = \sum_{j=1}^m r_j$ we have*

$$Trace(M^p_{r_1,\dots,r_m,\tau,s_1,\dots,s_m}(\Omega^m)) = M^p_{r,t,\tau m}(\Omega)$$

*for all $n,\ n/r > n_0$, where $n_0 = n_0(p,\tau,r_1,\dots,r_m,m)$.*

Very similar results with similar proofs on Traces of BMOA type spaces are valid also in bounded strongly pseudoconvex domains with smooth boundary. We refer readers to [15] for that paper. Note the unit ball and polyball cases which also have similar proofs were considered previously in [14].

Our proofs are based fully on properties of r lattices in the tube (see [4], [7], [12], [13]) and lemmas and estimates we presented above and are very similar to those provided earlier in the unit ball in the paper of first author and later in bounded pseudoconvex domains (see [14], [15]) based on properties of $r$-lattices in such domains (see [1], [2]).

We note hovewer structures of bounded strongly pseudovonvex and tubular domains are completely different.

**References**


1. *Abate M., Saracco A.* Carleson measures and uniformly discrete sequences in strongly pseudoconvex domains // Journal of the London Mathematical Society. 2011. Vol. 83, no. 3. P. 587–605.
2. *Abate M., Raissy J., Saracco A.* Toeplitz operators and Carleson measures in strongly pseudoconvex domains // Journal of Functional Analysis. 2012. Vol. 263, no. 11. P. 3449–3491.
3. *Andersson M., Carlsson H.* $Q_p$ spaces in strictly pseudoconvex domains // Journal d'Analyse Mathèmatique. 2001. Vol. 84, no. 1. P. 335–359.
4. *Bekolle D., Bonami A., Garrigos G., Nana C., Peloso M., Ricci F.* Lecture notes on Bergman projectors in tube domain over cones, an analytic and geometric viewpoint // Proceeding of the International Workshop on Classical Analysis. Yaounde. 2001.
5. *Debertol D.* Besov spaces and boundedness of weighted Bergman projections over symmetric tube domains // Publicacions Matemàtiques. 2005. Vol. 49, no. 1. P. 21–72.
6. *Djrbashian A. E., Shamoian F. A.* Topics in the theory of $A^p_\alpha$ spaces. Leipzig, Teubner, 1988.
7. *Faraut J., Koranyi A.* Analysis on symmetric cones. Oxford University Press, New York, 1994. xii + 382 p. ISBN: 0-19-853477-9.
8. *Ortega J., Fabrega J.* Mixed norm spaces and interpolation // Studia Matematica. 1994. Vol. 109, no. 3. P. 233–254.
9. *Rudin W.* Function theory in the polydisk. New York, 1969.
10. *Sehba B. F., Nana C.* Carleson Embeddings and two operators on Bergman spaces of tube domains over symmetric cones // Integral Equations and Operator Theory. 2015. Vol. 83, no. 2. P. 151–178.
11. *Sehba B.* Hankel operators on Bergman spaces of tube domains over symmetric cones // Integral Equations and Operator Theory. 2008. Vol. 62, no. 2. P. 233–245.
12. *Sehba B.* Operators in some analytic function spaces and their dyadic counterparts. PhD dissertation. University of Glasgow, 2009.
13. *Sehba B.* Bergman-type integral operators in tube domains over symmetric cones // Proceedings of the Edinburgh Mathematical Society, 2009. Vol. 52. P. 529–544.
14. *Shamoyan R., Mihić O.* On traces of $Q_p$ type spaces and mixed norm analytic function spaces on polyballs // Siauliai Mathematical Seminar. 2010. Vol. 5, no. 13. P. 101–119.
15. *Shamoyan R. F., Mihić O.* On a sharp trace theorem in BMOA type spaces in pseudoconvex domains // Comptes Rendus de L'Academie Bulgare des Sciences. 2017. Vol. 70, no. 2. P. 161–166.
16. *Xiao J.* Geometric $Q_P$ functions. Birkhäuser Verlag, 2006. 241 p.
17. *Zhu K.* Spaces of Holomorphic functions in the unit ball. Part of the *Graduate Texts in Mathematics* book series. Springer, New York, 2005. 274 p.

## MATHEMATICS

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# On traces of new BMOA type spaces in tubular domains over symmetric cones

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We provide new sharp trace theorems in new BMOA type analytic function spaces in tubular domains over symmetric cones under certain condition on the Bergman kernel. We obtain based on known properties of r-lattices in tubular domains complete analogues of our previously known results obtained previously by the authors in the unit ball and then in bounded strongly pseudoconvex domains with smooth boundary. Proofs of all theorems in all domains are based mainly on same ideas.

**Keywords:** *analytic function, BMOA type space, tubular domains, trace theorems.*

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