MSC 32A10; 32A37 Original article
DOI: https://doi.org/10.47928/1726-9946-2022-22-2-41-49

On some new sharp estimates of Toeplitz operator in some spaces of Hardy-Lizorkin type of analytic functions in the polydisk

Romi F. Shamoyan
Bryansk State Technical University, Bryansk, Russia
rshamoyan@gmail.com

Abstract. We provide some new sharp assertions on the action of Toeplitz $T\varphi$ operator in new $F_\alpha^{p,q}$ type spaces of analytic functions of several complex variables extending previously known assertions proved by various authors.

Keywords: polydisk, analytic function, Toeplitz operator, unit disk, Bergman spaces, Hardy spaces, Hardy-Lizorkin spaces

Acknowledgments: the author are thankful to the anonymous reviewer for his valuable remakes.

For citation. R. F. Shamoyan On some new sharp estimates of Toeplitz operator in some spaces of Hardy-Lizorkin type of analytic functions in the polydisk. Adyghe Int. Sci. J. 2022. Vol. 22, No. 2. P. 41–49. DOI: https://doi.org/10.47928/1726-9946-2022-22-2-41-49

УДК 517.55 Научная статья

О точных оценках для операторов Теплица в аналитических пространствах Харди-Лизоркина в единичном полидиске

Р. Ф. Шамоян
Брянский государственный технический университет, г. Брянск, Россия
rshamoyan@gmail.com

Аннотация. В заметке получены новые точные оценки для оператора Теплица, действующих в пространствах Харди-Лизоркина в единичном полидиске, ранее точные результаты такого рода были получены в менее общих функциональных пространствах в единичном круге.

Ключевые слова: аналитическая функция, полидиск, классы Харди и Бергмана, операторы Теплица, классы Харди-Лизоркина

Благодарности: автор выражает благодарность рецензентам за указанные замечания, которые позволили повысить качество статьи.

Для цитирования. Шамоян Р. Ф. О точных оценках для операторов Теплица в аналитических пространствах Харди-Лизоркина в единичном полидиске // Доклады АМАН. 2022. Т. 22, № 2. С. 41–49. DOI: https://doi.org/10.47928/1726-9946-2022-22-2-41-49

Introduction

Let U^n be the unit polydisk in $\mathbb{C}^n$, $U^n = \{z \in \mathbb{C}^n : |z| < 1, j = 1, ..., n\}$. Let further $H(U^n)$ be the space of all analytic functions in U^n. Let further also

$$F_\alpha^{p,q}(U^n) = \left\{ f \in H(U^n) : \|f\|^p_{F_\alpha^{p,q}} = \int_{T^n} \left(\int_{I^n} |f(r\xi)|^q (1-r)^{\alpha q - 1} dr \right)^{\frac{p}{q}} d\xi < \infty \right\},$$

where $0 < p, q < \infty, \alpha > 0, T^n = \{|z_j| = 1, j = 1, ..., n\}$, $I^n = (0,1] \times \cdots \times (0,1]$, $dr = dr_1 \cdots dr_n, d\xi = d\xi_1 \cdots d\xi_n$, $(1-r)^\alpha = \Pi_{k=1}^n (1-r_k)^\alpha$, $r_k \in (0,1)$ be the holomorphic Lizorkin-Triebel space, (see, for example, [1]-[5]). Note for particular case $p = q$ we have Bergman classical class, for $q = 2$ we have so-called Hardy-Lizorkin space H_β^p for some β that is, $H_\beta^p = \{f \in H(U^n) : D^\beta f \in H^p\}$, $0 < p \le \infty, \beta > 0$, where D^β is a fractional derivative of analytic f function in U^n. Note (see definitions bellow) for this particular cases the action of $T\varphi$ classical Toeplitz operator is well-studied in unit disk, unit ball and unit polydisk. We study T_φ operators in more general $F_\alpha^{p,q}$ type spaces in the polydisk. Our main sharp result provide some criteria for symbol of T, φ to obtain boundlessness of T_φ in mentioned type analytic spaces.

We define classical Hardy space $H^p(U^n)$, $0 < p \le \infty$ as follows (see also, for example, [1] and [6]). Let

$$H^p(U^n) = \{f \in H(U^n) : \|f\|_{H^p} = \sup_{r \in I^n} M_p(f, r) < \infty\},$$

and

$$M_p(f, r) = \left(\int_{T^n} |f(r\xi)|^p dm_n(\xi) \right)^{1/p},$$

where $r\xi = (r_1\xi_1, ..., r_n\xi_n)$ and dm_n is a normalized Lebesgues measure on T^n, $r_j \in (0,1)$, $j = 1, ..., n$. Note $M_p(f, r)$ function is growing function by each r_j and we, for $p = \infty$, obtain classical and well-studied class $H^\infty(U^n)$ of all bounded analytic functions in U^n (see for example [7] for this class of functions). Various sharp results on action of Toeplitz operators can bee seen in papers of various authors in various functional spaces in the unit ball and polydisk. We mention, for example, the following papers [8], [9], where such type sharp results can be seen in particular cases of $F_\alpha^{p,q}$ spaces namely in Bergman and in Hardy type spaces in the unit ball and in the unit polydisk. We note similar type results in particular values of parameters is well-known, (see, for example, [1], [4], [6], [8], [9]).

Such type sharp result on boundedness of Toeplitz operators also have various applications (see, for example [1], [4], [6], [9]).

We remind the reader the standard definition of Toeplitz T_h operators in the unit polydisk.

Let $h \in L^1(T^n)$. Then we define Toeplitz T_h operator as one integral operator

$$(T_h f)(z) = \frac{1}{(2\pi)^n} \int_{T^n} \frac{f(\xi_1, ..., \xi_n) h(\xi_1, ..., \xi_n)}{\Pi_{k=1}^n (1 - \bar{\xi}_k z_k)} d\xi_1 \cdot d\xi_n, \quad k = 1, ..., n, \; z_k \in U.$$

Note that we can easily show $F^{p,q}_{\alpha}$ general mixed norm analytic function spaces in the unit polydisk are Banach spaces for all values of p and q, if $\min(p,q) > 1$ and they are complete metric spaces for all other values of p and q.

We stress that behavior of the operators in the unit polydisk is substantially different from the action of $T\varphi$ operators in the unit ball in $\mathbb{C}^n$ (see [4], [6], [8], [9] for example). Our intention to set criteria for the action of Toeplitz T_φ operators from $F^{p,q}_{\alpha,k}(U^n)$ into Bergman-Sobolev and Hardy-Lizorkin type spaces in the unit polydisk, under the assumption that φ is holomorphic, $\varphi \in H(D^n)$ (some restriction on symbol of Toeplitz operator).

We define some new function spaces in the polydisk for formulation of our main result in the polydisk. Let further dm_{2n} be the normalized Lebesgues measure in $U^n, D^s, 0 < s \leq \infty$ be the fractional derivative of holomorphic f function

$$(D^s f)(z) = \sum_{|k| \geq 0} \frac{\Gamma(s+k+1)\Gamma(s+1)}{\Gamma(k+1)} a_k z^k,$$

$a_k z^k = a_{k_1 \ldots k_m} z_1^{k_1} \cdots z_m^{k_m}$, $f(z) = \sum\limits_{|k| \geq 0} a_k z^k, z \in U^n$, $\Gamma(\alpha+1) = \prod\limits_{j=1}^{m} \Gamma(\alpha_j + 1), \alpha_j > -1$, $j = 1, ..., m$.

Note if $f \in H(U^n)$ then for any $s \in N, D^s f \in H(U^n)$. Let also

$$F^{p,q}_{\alpha,k}(U^n) = \{f \in H(U^n) : \|D^k f\|_{F^{p,q}_{\alpha}} < \infty\}, 0 < p,q,\alpha < \infty,\ k \in \mathbb{N}.$$

$$A^s_{\alpha,m}(U^n) = F^{s,s}_{\alpha/s,m} = \left\{ f \in H(U^n) : \|f\|^s_{A_{\alpha,m}} = \int_{U^n} |(D^m f)(z)|^s (1-|z|)^{\alpha-1} dm_{2n}(z) < \infty \right\},$$

$m \in \mathbb{N}, 0 < s, \alpha < \infty$ (Bergman-Sobolev space). Let further

$$H^s_m(U^n) = \{f \in H(U^n) : \|D^m f\|_{H^s} < \infty,\ m \in \mathbb{N},\ 0 < s < \infty\}$$

be analytic Hardy-Lizorkin space in the unit polydisk U^n.

Note it can easily shown that these both scales of analytic function spaces in the unit polydisk are Banach spaces for all values of $s, s \geq 1$ and they are complete metric spaces for other values of $s, s > 0$.

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

Main results

We prove our main sharp theorems in this section.

Note that for p=q theorems 1,2 in the unit disk are probably well-known for experts (see for example [4],[6],[8]] and various references there).

Lemma 1. *Let $f \in F^{p,q}_{\alpha,k}(U^n)$, $s > 1$, $k \in N$, $0 < p, q \leq s$ and $k = \frac{(\alpha+\frac{1}{p})s-1}{s} + m\left(1-\frac{1}{s}\right)$. Then the following equality is valid*

$$\left(\int_{U^n} |D^k f(w)|^s (1-|w|)^{s(k-1)-(m-1)(1-\frac{1}{s})} dm_{2n}(w)\right)^{\frac{1}{s}} \leq$$

$$\leq c \left(\int_{T^n} \left(\int_{I^n} |D^k f(w)|^q (1-|w|)^{\alpha q-1} d|w|\right)^{\frac{p}{q}} dm_n(\xi)\right)^{\frac{1}{p}}.$$

Theorem 1. *Let* $0 < \max(p,q) \le s,\ 1 < s < \infty,\ k = \alpha + \frac{1}{p} - \frac{1}{s} + m\left(1 - \frac{1}{s}\right),\ m, k \in \mathbb{N}, \alpha > 0.$ *Then* $T_{\bar{\varphi}}$ *operator is bounded operator from* $F^{p,q}_{\alpha,k}(U^n)$ *into* $A^s_{m,m}(U^n)$ *if and only if* $\varphi \in H^\infty(U^n)$ *and* $\|\varphi\|_\infty \le \|T_{\bar{\varphi}}\|$.

Proof. Note if $f_r(z_1, ..., z_n) = \prod\limits_{i=1}^{m} \frac{1}{1 - r_j z_j},\ z_j \in U, r_j \in (0,1), j = 1, ..., n.$ Then

$$T_{\bar{h}}(f_r) = \frac{1}{(2\pi i)^n} \int_{T^n} \frac{f_r(\xi_1, ..., \xi_n)\overline{h(\xi_1, ..., \xi_n)}}{\prod\limits_{j=1}^{n}(\xi_j - z_j)} d\xi_1 ... d\xi_n = \frac{1}{(2\pi i)^n} \int_{T^n} \frac{\overline{h(\xi_1, ..., \xi_n)} d\xi_1 ... d\xi_n}{\prod\limits_{j=1}^{n}(\xi_1 - z_j)(1 - r_j \xi_j)} =$$

$$= \frac{(-1)^n}{(2\pi i)^n} \int_{T^n} \frac{h(\xi_1, ..., \xi_n) d\bar{\xi}_1 ... d\bar{\xi}_n}{\prod\limits_{j=1}^{n}(\bar{\xi}_j - \bar{z}_j)(1 - r_j \bar{\xi}_j)} = \frac{1}{(2\pi i)^n} \int_{T^n} \frac{h(\xi_1, ..., \xi_n) d\xi_1 ... d\xi_n}{\prod\limits_{j=1}^{n}(\bar{\xi}_1 - \bar{z}_j)\xi_j^2(1 - r_j \bar{\xi}_j)} =$$

$$= \frac{1}{(2\pi i)^n} \int_{T^n} \frac{h(\xi_1, ..., \xi_n) d\xi_1 ... d\xi_n}{\prod\limits_{j=1}^{n}(1 - \bar{z}_j \xi_j)(\xi_j - r_j)} = \overline{\frac{\overline{h(r_1, ..., r_n)}}{\prod\limits_{j=1}^{n}(1 - \bar{z}_j r_j)}} = \frac{\overline{h(r_1, ..., r_n)}}{\prod\limits_{j=1}^{n}(1 - z_j r_j)},\ z_j \in U^n, j = 1, ..., m.$$

Using these equalities we can show one part of our theorem (necessity of condition on symbol).

The necessity of condition $\varphi \in H^\infty(U^n)$ can be shown in a standard way. Let $T_{\bar{\varphi}}$ be a bounded operator from $F^{p,q}_{\alpha,k}(U^n)$ into $A^s_{m,m}(U^n)$. We consider the following test function:

$$f_r(z) = \frac{(1 - r)^{k + 1 - \alpha - \frac{1}{p}}}{(1 - rz)},$$

where $(1 - rz) = \prod\limits_{j=1}^{n}(1 - r_j z_j),\ r \in I^n,\ z \in U^n, r_j > \frac{1}{2}, j = 1, ..., n.$ Note it is easy to check using well-known Forelly-Rudin type estimates and the chain of equalities we provided in the start of our proof that the following two estimates are valid.

$$\|f_r(z)\|_{F^{p,q}_{\alpha,k}} \le C,\ k > \frac{1}{p} + \alpha - 1, \tag{1}$$

$$\|T_{\bar{\varphi}}(f_r)(z)\|_{A^s_{m,m}} = r|\varphi(r)| \|f_r(z)\|_{A^s_{m,m}} \ge r|\varphi(r)|C',\ r \in (C, 1). \tag{2}$$

Namely we use the following standard equation to get (1), (2)

$$D^\alpha\left(\frac{1}{1 - rz}\right) = \frac{1}{(1 - rs)^{\alpha+1}},\ \alpha > 0, r \in (0,1), z \in U^n,$$

and the estimates (see [1], [3]-[6])

$$\int_{U^n} \frac{(1 - |z|)^t dm_{2n}(z)}{|1 - \bar{z}w|^v} \asymp \frac{1}{(1 - |w|)^{v - t - 2}}, w \in U^n, t > -1, v > t + 2,$$

where $|1 - zw|^\alpha = \prod\limits_{j=1}^{n} |1 - z_j w_j|^\alpha,\ z, w \in U^n, \alpha \ge 0,$

$$\int_{T^n} \frac{dm_n(\xi)}{|(1 - rz)^\alpha|} \asymp \frac{1}{(1 - rR)^{\alpha - 1}}, \alpha > 1, r, R \in (0,1), z = R\xi, z \in U^n.$$

Hence from (1) and (2) we obtain directly that $|\varphi(r)| \leq C', r \in I^n$, from some positive constant C'.

Note for $|\varphi(z)| \leq C'', z \in U^n$, we must consider $\tilde{f}_r(z) = f_r(e^{-i\theta}z),\ \theta \in T^n,\ r \in (0,1)$, $e^{-i\theta}z = (e^{-i\theta_1}z_1, ..., e^{-i\theta_n}z_n)$. The main problem is now to show the reverse (sufficiency condition symbol).

Let $\varphi \in H^\infty(U^n)$. Let $F(w) = T_{\bar{\varphi}}f(w), R \in (0,1)$. Then using standard duality arguments it is easy to show that

$$\|F_R(w)\|^s_{A^s_{m,m}} = \int_{U^n} |D^m F_R(w)|^s (1-|w|)^{m-1} dm_{2n}(w) =$$

$$= \left| \int_{U^n} D^m F_R(w) \overline{G_R(w)} (1-|w|)^{m-1} dm_{2n}(w) \right|,$$

where $G_r(w) \in L^{s'}_m$, $\|f\|^{s'}_{L^{s'}_m} = \int_{U^n} |f(z)|^{s'} (1-|z|)^{m-1} dm_{2n}(z) < \infty$, $\frac{1}{s} + \frac{1}{s'} = 1$, and $\|G_R(w)\|_{L^{s'}_m} = 1$. Note further $F_R(w) = (T_{\bar{\varphi}}f)(Rw) = \frac{1}{(2\pi)^n} \int \frac{\bar{\varphi}(t)f(t)}{1-\bar{t}Rw} dm_n(t)$. And hence we have

$$\|F_R(w)\|_{A^s_{m,m}} = C(n) \left| \int_{U^n} \int_{T^n} \frac{\overline{\varphi(t)} f(t) \overline{G_R(w)} (1-|w|)^{m-1}}{(1-\bar{t}Rw)^{m+1}} dm_{2n}(w) dm_n(t) \right|.$$

Using Fubinis theorem we note that the following map

$$S(G_R) = \Phi(\bar{t}R) = \int_{U^n} \frac{(\overline{G_R(w)})(1-|w|)^{m-1}}{(1-\bar{t}Rw)^{m+1}} dm_{2n}(w) =$$

$$= \int_{U^n} \frac{(\overline{G_R(z)})(1-|z|)^{m-1}}{(1-\bar{t}Rz)^{m+1}} dm_{2n}(\bar{z}),\ z = \bar{w},$$

is a Bergman projection map acting from $L^{s'}_m$ into $A^{s'}_m$, $A^{s'}_m = L^{s'}_m \cap H(U^n), \frac{1}{s} + \frac{1}{s'} = 1$. This important known fact have many applications and can be seen in [3], [4], [6], [7]. Using Littlewood-Paley equality in the unit polydisk with D^α operator namely the following equality

$$\frac{1}{(2\pi)^n} \int_{T^n} f(rt) g(r\bar{t}) dm_n(t) =$$

$$= \left(\frac{m}{\pi}\right)^n \prod_{j=1}^n (r_j^{-2m}) \int_0^{r_1} \cdots \int_0^{r_n} \int_{T^n} (D^m g(R\xi))(f(R\bar{\xi})) \prod_{i=1}^n (r_i^2 - R_i^2)^{m-1} R dR dm_n(\xi), \qquad (3)$$

$f, g \in H(U^n), r \in I^n, m \in \mathbb{N}$ (see [1], [4]-[7], [9], [10]).

Hence we have using this equality and Hölder's inequality in the unit polydisk

$$\|F_R(w)\|_{A^s_{m,m}(U^n)} = C(n) \left| \int_{T^n} \phi(\bar{t}R) f(t) \overline{\varphi(t)} dm_n(t) \right|,\ \phi(w) \in A^{s'}_m.$$

Then we have the following

$$\|F(w)\|_{A^s_{m,m}(U^n)} = \lim_{R\to1} \|F_R(w)\|_{A^s_{m,m}(U^n)} = C(n) \left| \lim_{R\to1} \int_{T^n} \phi(\bar{t}R)(f(t)) \overline{\varphi(t)} dm_n(t) \right| =$$

$$= C(n)\left|\lim_{R\to 1}\int_{T^n}\phi(\bar{t}R)f(Rt)\bar{\varphi}(\bar{t}R)dm_n(t)\right|.$$

Using (3) we have

$$\|F(w)\|_{A^s_{m,m}(U^n)} \le \tilde{c}\int_{U^n}|D^k f(\bar{w})|(1-|w|)^{k-1}|\bar{\varphi}(w)||\phi(w)|dm_{2n}(w) \le$$

$$\le \tilde{\tilde{c}}\|\varphi\|_\infty\int_{U^n}|D^k f(w)|(1-|w|)^{k-1}|\phi(w)|dm_{2n}(w).$$

From this estimate we using Hölder's inequality have now

$$\int_{U^n}|D^k f(w)|(1-|w|)^{k-1}|\phi(w)|dm_{2n}(w) \le$$

$$\le \left(\int_{U^n}|D^k f(w)|^s(1-|w|)^{(k-1)s-\frac{s}{s'}(m-1)}dm_{2n}(w)\right)^{\frac{1}{s}}\left(\int_{U^n}(1-|w|)^{m-1}|\phi(w)|^{s'}dm_{2n}(w)\right)^{\frac{1}{s'}},$$

$\frac{1}{s}+\frac{1}{s'}=1$.

It remains to use the following lemma (see [8]) for $F^{p,q}_{\alpha,k}$ type analytic function spaces. Theorem 1 is proved.

Remark 1. Note similar proof can be provided for a little bit general situation where standard $(1-|w|)^\alpha$ type weights are replaced with $w(r), r\in(0,1)$ type weights (see for these weights, for example, [4], [8]).

Assume now that the following lemma is valid (see the proof below).

Lemma A. *Let $G\in H(U^n), 1<s<\infty$. Then the following estimates are valid*

$$M_s(D^m G, R^2)\le c(1-R)^{-m}M_s(G,R), R\in I^n, m\in\mathbb{N}$$

$$M_s(G,R^2)\le c_1(1-R)^{\frac{1}{s}-1}M_1(G,R), R\in I^n.$$

Proof. The second part follows directly from Lemma 2 (see [11], [12]), the first part is easy to get using Fubinis theorem and induction.

Remark 2. Note it is well known that those estimates are valid for $n=1$ case (see [1], [3], [4], [9]).

Theorem 2. *Let $0<\max(p,q)\le 1,\ 1<s<\infty,\ k=\frac{1}{p}-\frac{1}{s}+\alpha+m,\ m,k\in\mathbb{N},\alpha>0$. Then $T_{\bar{\varphi}}$ operator is bounded operator from $F^{p,q}_{\alpha,k}(U^n)$ into $H^s_m(U^n)$ if and only if $\varphi\in H^\infty(U^n)$ and $\|\varphi\|_\infty\le\|T_{\bar{\varphi}}\|$.*

Proof. The necessity of the $\varphi\in H^\infty(U^n)$ condition can be shown as in our previous theorem. It is enough to consider the following test function

$$f_r(z)=\frac{(1-r)^{k+1-\alpha-\frac{1}{p}}}{1-rz},$$

$(1-rz)=\prod_{i=1}^{n}(1-r_iz_i), r\in I^n, z\in U^n, k>\alpha+\frac{1}{p}-1$ and note in addition that $\|f_r(z)\|_{H^s_m}\ge C$ for $k-m=\frac{1}{p}-\frac{1}{s}+\alpha$. We show now that $\varphi\in H^\infty(U^n)$ condition is also sufficient.

Let $G(z)=(T\bar{\varphi}f)(z), R\in(0,1)$. Then using standard duality arguments we have the following equality

$$\|D^m G_R\|_{H^s(U^n)}=\left(\int_{T^n}|D^m G(Re^{i\varphi})|^s dm_n(\varphi)|\right)^{\frac{1}{s}}=$$

$$\left|\int_{T^n} D^m G(Re^{i\varphi})\overline{\phi(e^{i\varphi})}dm_n(\varphi)\right|, R \in (0,1), \phi \in L^{s'}(T^n).$$

Note then that $G(z) = T_{\bar{\varphi}}(f)(z) = \frac{1}{(2\pi i)^n}\int_{T^n} \frac{f(t)\overline{\varphi(t)}}{\prod\limits_{j=1}^{n}(1-\bar{t}_j z_j)}dm_n(t), \varphi \in H^\infty(U^n), z = Re^{i\varphi}$, we have that

$$D^m G(z) = \frac{1}{(2\pi i)^n}\int_{T^n} \frac{f(t)\overline{\varphi(t)}}{(1-\bar{t}z)^{m+1}}dm_n(t),\ z = Re^{i\varphi}.$$

And hence we have the following

$$\|D^m G\|_{H^s(U^n)} = \lim_{R\to 1}\|D^m G_R\|_{H^s(U^n)} =$$

$$= C(n)\left|\lim_{R\to 1}\int_{T^n}\int_{T^n}\overline{\phi(e^{i\psi})}\frac{f(t)\overline{\varphi(t)}}{(1-\bar{t}Re^{i\psi})^{m+1}}dm_n(t)dm_n(\psi)\right| =$$

$$= C(n)\left|\lim_{R\to 1}\int_{T^n} D^m\left(\int_{T^n}\frac{\overline{\phi(e^{i\psi})}dm_n(\psi)}{(1-\bar{t}Re^{i\psi})}\right)f(t)\bar{\varphi}(t)dm_n(t)\right|.$$

Note now that the following operator (see [4], [9])

$$(S(\phi))(\bar{t}R) = h(\bar{t}R) = \int_{T^n}\frac{\bar{\phi}(e^{-i\psi})dm_n(\psi)}{1-\bar{t}Re^{i\psi}}$$

is a Riesz projection of $\bar{\phi}$ function and hence $h(\bar{t}R) \in H^{s'}(T^n), \frac{1}{s}+\frac{1}{s'} = 1$. This important fact concerning boundedness of Riesz projection in Hardy spaces in polydisks can be seen in [7]. Hence by Littlewood-Paley identity (see above (3)) we have the following inequalities

$$\|D^m G\|_{H^s(U^n)} = C(n)\left|\lim_{R\to 1}\int_{T^n} f(Rt)\bar{\varphi}(R\bar{t})D^m h(\bar{t}R)dm_n(t)\right| \le$$

$$\le C_1(n)\int_{U^n}|D^k f(\bar{w})|(1-|w|)^{k-1}|\bar{\varphi}(w)|D^m h(w)|dm_{2n}(w) \le$$

$$\le C_1(n)\left(\int_{U^n}|D^k f(\bar{w})|(1-|w|)^{k-1}|D^m h(\bar{w})|dm_{2n}(w)\right)\|\varphi\|_\infty,$$

$w = r\xi, h(w) \in H^{s'}(T^n), \frac{1}{s}+\frac{1}{s'} = 1.$

Using Hölders inequality we finally have

$$\|D^m G\|_{H^s(U^n)} \le C_1(n)\|\varphi\|_\infty\int_{I^n} M_s(D^k f,\rho)M_{s'}(D^m h,\rho)(1-\rho)^{k-1}\rho d\rho_1...d\rho_n.$$

Now using lemma A from last estimates we obtained above, we get the following estimates for the $\|D^m G\|_{H^s(U^n)}$. We have

$$\|D^m G\|_{H^s(U^n)} \le c\|\varphi\|_\infty\|h\|_{H^{s'}(U^n)}\int_{I^n} M_s(D^k f,\rho)(1-\rho)^{k-m-1}\rho d\rho_1...d\rho_n,$$

$k > m, k, m \in \mathbb{N}$. Next, from Lemma B and Lemma 2 (see below) and Lemma A we have the following estimates

$$\int_{U^n}|G(w)|(1-|w|)^{\frac{1}{p}+\alpha-2}dm_{2n}(w) \le$$

$$\leq \left(\int_{T^n}\left(\int_{I^n}(1-R)^{\alpha q-1}|G(R\xi)|^q dR\right)^{\frac{p}{q}} dm_n(\xi)\right)^{\frac{1}{p}}, G\in H(U^n), \max(p,q)\leq 1, 0<\alpha<\infty,$$

$$\int_{I^n} M_s(G,R)(1-R)^{\beta} RdR \leq c\int_{U^n}|G(w)|(1-|w|)^{\frac{1}{s}-1+\beta} dm_{2n}(w), G\in H(U^n), \beta>-1, s>1.$$

Using this last two estimates we have

$$\|D^m G\|_{H^s(U^n)} \leq c\|\varphi\|_{\infty}\|h\|_{H^{s'}(U^n)}\|f\|_{F^{p,q}_{k,\alpha}} < \infty.$$

Theorem 2 is proved.

Lemma 2. *(see [11], [12]) Let f be analytic in $0\leq r_j<|z_j|<R_j, 1\leq j\leq m$ and $f\in C^s$ continuous in closure of this domain. Then for $0\leq p\leq q\leq\infty,\ \rho\in(r,R)$*

$$\|f_\rho\|_{H^q} \leq c(m,p,q)\prod_{j=1}^{m}\left((\rho_j-r_j),(R_j-\rho_j)\right)^{\frac{1}{q}-\frac{1}{p}} \max_{\substack{V_j=r_j,\ R_j\\ j=1,\ldots,m}} \|f_v\|_{H^p}.$$

Lemma B. *(see [4], [5]) Let $0<\max(p,q)\leq s<\infty, \alpha>0$. Then*

$$\left(\int_{U^n}|f(w)|^s(1-|w|)^{s\left(\alpha+\frac{1}{p}\right)-2} dm_{2n}(w)\right)^{\frac{1}{s}} \leq c\|f\|_{F^{p,q}_{\alpha}}.$$

Remark 3. It is easy to note that the same approach can be used to get criteria on symbol φ, for which $(T_{\bar{\varphi}})$ operator is bounded from $F^{p,q}_{k,\alpha}$ into X, where X is a different from A^m_s and H^s quazinormed subspace of $H(U^n)$.

We regret to inform that Dr. Mihic declined my suggestion to cooperate around this interesting paper.

References

1. *A. B. Aleksandrov, V. V. Peller* Hankel operators and similarity to a contraction. International Mathematics Research Notices. 1996. No. 6. P. 263–275.
2. *M. Arsenović, R. F. Shamoyan* On Multipliers on Holomorphic $F^{p,q}_{\alpha}$ type spaces on the unit polydisc. Journal of Siberian Federal University, Mathematics and Physics. 2012. Vol. 5, No. 4. P. 471–479.
3. *J. Ortega, J. Fábrega* Holomorphic Triebel-Lizorkin spaces. Funct. Anal. 1997. Vol. 151, No. 1. P. 177–212.
4. *R. F. Shamoyan* Multipliers, Toeplitz operators and duality in some spaces of analytic functions in the unit polydisk. PhD Dissertation. Moscow, 2001.
5. *R. F. Shamoyan* On multipliers from Bergman-type spaces to Hardy spaces in the polydisk. Ukrainian Math. J. 2000. Vol. 10. P. 1405–1415.
6. *A. B. Aleksandrov* Function Theory in the Ball, in Several Complex Variables II. Springer-Verlag, New York, 1994.
7. *W. Rudin* Function theory in the polydisks. Benjamin. New York, 1969.
8. *A. Harutyunyan, F. A. Shamoyan* Toeplitz operators in multidimensional spaces $H^p(\alpha)$ of M. M. Djrbashian. J. of Contemp. Math. An., National Ac. of Sci. of Armenia. 1995. Vol. 30, No. 2. P. 70–78.
9. *S. Janson, J. Peetre, S. Semmes* On the action of Hankel and Toeplitz operators on some function spaces. Duke Math. J. 1984. Vol. 51, No. 4. P. 937–958.
10. *W. Rudin* Function theory in the unit ball of $\mathbb{C}^n$. Springer-Verlag, New York, 1980.

11. *R. M. Trigub* Multipliers of class $H^p(D^m)$ for $p \in (0,1)$ and the approximative properties of summation methods for power series. Russian Acad. Sci. Dokl. Math. 1994. Vol. 335, No. 6. P. 697–699.
12. *R. M. Trigub* Multipliers in the Hardy spaces $H^p(D^m)$ for $p \in (0,1]$ and the approximative properties of summability methods for power series. Mat. Sbornik. 1994. Vol. 188, No. 4. P. 145–160.

Поступила 06.05.2022; одобрена после рецензирования 06.06.2022; принята к публикации 29.06.2022.

Submitted 06.05.2022; approved after reviewing 06.06.2022; accepted for publication 29.06.2022.

Об авторе:

Шамоян Роми Файзович, кандидат физико-математических наук, Брянский государственный технический университет, (г. Брянск, Россия), ORCID: https://orcid.org/0000-0002-8415-9822, rsham@mail.ru

About the author:

Romi Fayzovich Shamoyan, Ph.D. (Phys. & Math.), Senior Researcher, Bryansk State Technical University, (Bryansk, Russia), ORCID: https://orcid.org/0000-0002-8415-9822, rsham@mail.ru

Автор прочитал и одобрил окончательный вариант рукописи.

The author has read and approved the final version of the manuscript.