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On the action of area integral on product domains and bounded functionals in Hardy type spaces

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Abstract. We provide a new maximal theorem for new mixed norm Hardy type spaces and an extention of Hardy-Littlewood theorem to such type mixed norm function spaces related with Area integral on product domains, some new duality results on bounded functionals for such type mixed norm Hardy spaces in product domains will be also provided. We consider and discuss also many new function spaces with mixed norm.

Keywords: Bounded linear functionals, Hardy spaces, area integral, product domains, maximal theorem, Hardy-Littlewood type theorem

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УДК 517.55 Научная статья

О действии интеграла площадей на область произведения и ограниченные функционалы в пространствах типа Харди

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Аннотация. В заметке предпринята попытка с помощью наложения ограничений распространить некоторые результаты Натальи Часовой о классах типа Харди в шар из полидиска. Новые теоремы похожего типа будут также обсуждаться. Приведена новая максимальная теорема и теоремы о представлении ограниченных непрерывных функционалов. вводятся различные шкалы новых функцональных пространств типа Харди со смешанной нормой.

Ключевые слова: Шар, класс Харди, полидиск, максимальная теорема аналитическая функция, ограниченный функционал

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The main intention of this note to generalize some known theorems from [1] on new mixed norm Analytic Hardy function spaces from the case of unit polydisk to the case of more general products of unit balls.

We obtain all our results in this note by carefully analysing known proofs of less general case.

Practically everywhere complete generalizations by simple repetition of all arguments of less general case from [1] were found by us. In one assertion below concerning linear bounded functionals an additional condition which we put also provided such type interesting new generalization.

Note that all our results are well-known in the unit polydisk, unit disk (see for this cases, for example, [1] for mixed norm Hardy spaces)and for those even more particular cases when all p_j parameters are equal to each other.in the unit polydisk (see, for example, [3] for interesting old results on bounded linear functionals in classical Hardy classes in the unit polydisk).

Analytic Hardy spaces or Hardy type function spaces of one or several complex variables is a classical topic of complex function theory, many results and their various interesting applications in one dimension and also in higher dimension were given previously by many authors and are well-known (see, for example, [1] and various referenes there).

For formulation of our main results we need some standard definitions.of analytic complex function theory in the unit ball.

Let B^n be the unit ball in $\mathbb{C}^n, S_n$ be the unit sphere we define certain new Hardy type function spaces in $\mathbb{C}^n$ as follows (similarly in pseudoconvex domains).

Let further.

$$A_\infty H^{\vec{p}} = \{f \in H(B^n \times \cdots \times B^n) \colon$$
$$\left(\int\limits_{S^n} \sup_{r_m<1} \cdot \int\limits_{S^n} \sup_{r_2<1} \left(\int\limits_{S^n} \sup_{r_1<1} |f(\vec{r}\xi)|^{p_1} \, d\xi_1 \right)^{\frac{p_2}{p_1}} \dots d(\xi_m) \right)^{\frac{1}{p_m}} < \infty\};$$

$p_i < \infty, i = 1, \dots, m;$

$$M_\infty(H^{\vec{p}}) = \{f \in H(B \times \cdots \times B) \colon$$
$$\left(\int\limits_{S^n} \dots \int\limits_{S^n} \sup_{z_m \in \Gamma_t(\xi)} \left(\sup_{z_1 \in \Gamma_t(\xi)} |f(z_1, \dots, z_m)|^{p_1} \, d\xi_1 \dots d\xi_m \right)^{p_1} \dots d(\xi_m) \right)^{\frac{1}{p_m}} < \infty\};$$

$p_i < \infty, i = 1, \ldots, m$; where $\Gamma_t(\xi) = \{z \in B\colon |1 - \bar{z}\xi| < t(1 - |z|)\}, t > 1, \xi \in S_n$ and $H(B \times \cdots \times B)$ - is a space of all analytic functions on $B \times \cdots \times B$ (by each variable) and we define also two other similar type functions spaces of Hardy type $A_\infty \tilde{H}^{\vec{p}}$ and $M_\infty \tilde{H}^{\vec{p}}$ using same type quazinorms with expressions

$$\sup_{z_j \in \Gamma_t(\xi), j=1,\ldots,m} |f(z_1, \ldots, z_m)|$$

and also

$$\sup_{z_j < 1, j=1,\ldots,m} |f(\vec{z}\xi)|;$$

within the quazinorm we use.

Note that if

$$f(\vec{z}) = \prod_{i=1}^{m} f_i(z_i)$$

then

$$\|f\|_{A_\infty H^{\vec{p}}} = \prod_{i=1}^{m} \|f_i\|_{A_\infty H^{p_i}};$$

and the same for other spaces ($M_\infty H^{\vec{p}}$).

Let $0 < p_i < \infty; i = 1, \ldots, m$. Let also

$$H^{\vec{p}} = \left\{ f \in H(B_n \times \cdots \times B_n)\colon \left(\sup_{r<1}\right) \left(\int_{S^n} \cdots \left(\int_{S^n} |f(\vec{r}\xi)|^{p_1} \, d\xi \right)^{\frac{p_2}{p_1}} \cdots \right)^{\frac{1}{p_m}} < \infty \right\}.$$

We formulate first several extensions of classical results on these new function spaces, which extend some well known assertions We denote by D^α the well known fractional derivative of f function below.

Theorem 1. Let $\psi \in L^{\vec{q}}(S^n \times \cdots \times S^n), let 1 < q_i < \infty; \frac{1}{p_i} + \frac{1}{q_i} = 1; i = 1, \ldots, m.$

Let

$$g(z_1, \ldots, z_n) = \int_{S^n} \cdots \int_{S^n} \frac{\psi(\xi_1, \ldots, \xi_m)}{\prod_{i=1}^{m}(1 - \bar{\xi}_i z_i)} d\vec{\xi} \in H^{\vec{q}}(S \times \cdots \times S)$$

Then we have the following equality $(H^{\vec{p}})^* = H^{\vec{q}}$ and bounded functionals of $H^{\vec{p}}$ represented by Cauchy standed duality as

$$(\Phi)(f) = (\lim_{r \to 1}) \int_{S^n} \cdots \int_{S^n} f(rt) \cdot g(r\vec{t}\,) dt; g \in H^{\vec{q}}.$$

Theorem 2. Assume that

$$\underbrace{\int_{S^n} \cdots \int_{S^n}}_{m} \|f(z_1, \ldots, z_m)\| \times \prod_{j=1}^{m} (1 - |z|)^{\frac{1}{p_j} - (n+1)} dm(\vec{z}) \leq c \|f\|_{H^{\vec{p}}}; p_j < 1, j = 1, \ldots, m.$$

Then $(H^{\vec{p}})^* = \left\{ f \in H(B \times \dots \times B)\colon \left| D_{z_j}^{\vec{\alpha}+1} g(z_1, \dots, z_m) \right| \prod_{j=1}^{m} (1 - |z_j|)^{\alpha_j - \frac{1}{p_j} + n + 1} < \infty \right\}$;

Remark. This result of Theorem 2 is also valid if we replace $H^{\vec{p}}$ by $A_\infty H^{\vec{p}}$ or $M_\infty H^{\vec{p}}$ or $A_\infty \tilde{H}^{\vec{p}}$ or $\widetilde{M_\infty} H^{\vec{p}}$.

Let further P be Riesz projection on S^n.

Theorem 3.

1) Assume $\|P_\xi(g)\|_{L^{\vec{p}}} \le \tilde{c} \|g\|_{L^{\vec{p_j}}}; p_j > 1; m > n + 1$. Then

$$\left\| \int\limits_{\Gamma_t(\xi_m)} \dots \int\limits_{\Gamma_t(\xi_1)} \prod_{j=1}^{m} (1 - |z_j|)^{m-(n+1)} \cdot \left| D_{\vec{z}}^m \left(f\left(\vec{z} \right) \right) \right| dv(z) \right\|_{L^{\vec{p}}} \le c \|f\|_{H^{\vec{p}}}.$$

2) We also have $0 < p_i < \infty, i = 1, \dots, m$;

$$\left(\int\limits_{S^n} \dots \sup_{\Gamma_t(\xi_m)} \int\limits_{S^n} \sup_{\Gamma_t(\xi_m)} |f(z)|^{p_1} \right)^{\frac{1}{p_m}} \le c \|f\|_{H^{\vec{p}}}.$$

We need some lemmas.

Lemma A. ([2], [3], [4], [5]) Let $0 < p_i < \infty, \alpha_i > -1, i = 1, \dots, n$. Let $F \in A^{\vec{p}}(\vec{\alpha})$. Then Φ_β function

$$\Phi_\beta(z_{m+1}, \dots, z_n) = \|F\|^\beta_{A^{p_1, \dots, p_m}(\alpha_1, \dots, \alpha_m)},$$

for $(z_{m+1}, \dots, z_n) \in B^{n-m} = \underbrace{B \times \dots \times B}_{n-m}$ is a subharmonic ($(n-m)$ subharmonic) in B^{n-m} for all $\beta \ge 0$.

Lemma B. (See [6]).

$$\int\limits_S \sup_{z_1 \in S_\alpha(\xi_1)} |f(z_1, \cdot)|^{p_1} d\xi_1 \le c \int\limits_S |f(\xi_1, \cdot)|^{p_1} d\xi_1$$

for all $0 < p_1 < \infty, \alpha > 0, S_\alpha(\xi_1) = \{z \in B\colon |1 - \xi_1 z| < \alpha(1 - |z|)\}$. The same is valid if we replace $S_\alpha(\xi)$ by $0 < r < 1$ (see [6]).

Some final remarks. Similar results are valid for mixed cases.We provide some examples. Let $B(z, r)$ be standart Bergman ball in the unit ball B (see [6]).

$$\int\limits_B \left(\sup_{z \in B(w,r)} \right) |f(z)|^p dV_\alpha(w) \le c \|f\|^p_{A^p_\alpha} \tag{A}$$

for $0 < p < \infty; \alpha > -1$; so

$$\int\limits_S \sup_{z_2 \in S_\alpha(\xi_2)} \left(\int\limits_B \sup_{z \in B_1(w,r)} |f(\vec{z})|^{p_1} dV_\alpha(w) \right)^{\frac{p_2}{p_1}} d\xi_2 \le c \int\limits_S \left(\int\limits_B |f(\vec{z})|^{p_1} dV_\alpha(w) \right)^{\frac{p_2}{p_1}} d\xi_2,$$

$0 < p_i < \infty, i = 1, 2, \alpha > -1$, and for more general cases for functions of m-variable. We refer for (A) to [2].

We can consider various combinations of these two maximal theorems for functions of m-variables integrating z_j by S, then z_i by B. It is an open problem to prove such type assertions for more complicated domains as tubular domains over symmetric cones and for bounded strongly pseudoconvex domains in C^n.

Note very similar proof is valid for another maximal theorem

$$\left(\int\limits_{S^n} \sup_{0<r_n<1} \dots \left(\sup_{0<r_1<1} |f(\vec{z})|^{p_1} \, d\xi_1\right)^{\frac{p_2}{p_1}} \dots d\xi_n\right)^{\frac{1}{p_n}} \leq \tilde{c} \, \|f(z)\|_{H^{\vec{p}}},$$

$0 < p_j < \infty, j = 1, \dots, n$ for mixed Hardy type spaces.

We consider two scales of analytic function spaces of Hardy-Bergman type with mixed norm in polyball $B \times \dots \times B$

$$\sup_{\substack{0<r_j<1\\ j=1,\dots,m}} \left(\int\limits_S \dots \left(\int\limits_S |f(\vec{r}\xi)|^{p_1} \, d\xi_1\right)^{\frac{p_2}{p_1}} \dots d\xi_m\right)^{\frac{1}{p_m}} < \infty,$$

and changing integration by ball mixing this quazinorm with the Bergman space we arrive at

$$\sup_{\substack{0<r_j<1\\ j=i_1,\dots,i_m}} \left(\int\limits_B (1-|z|)^{\alpha_1} dV(z_1) \int\limits_S \dots \int\limits_B (1-|z_j|)^{\alpha_j} dV(z_j) \times \right.$$
$$\left. \times \left(\int\limits_S |f(\vec{r}\xi)|^{p_1} \, d\xi_1\right)^{\frac{p_2}{p_1}} \dots \right)^{\frac{1}{p_m}} < \infty;$$

$\alpha_k > -1, k = 1, \dots, j$; for $p_i \leq 1, i = 1, \dots, m$,we can give some results related with the dual spaces in both scales. Note we can also consider the following Hardy type spaces.

$$\left[\sup_{z_2 \in \Gamma_{\alpha_2}(\xi_2)} \left(\int\limits_S \left(\sup_{z_1 \in \Gamma_{\alpha_1}(\xi_1)}\right) \int\limits_S |f(\vec{r}\xi)|^{p_1} \, d\xi_1\right)\right]^{\frac{1}{p_m}} < \infty,$$

and

$$\sup \int\limits_S \left(\sup_{r_m<1} \int\limits_S \left(\sup_{r_1<1} |f(r\xi)|^{p_1}\right)^{\frac{p_2}{p_1}}\right)^{\frac{1}{p_m}} < \infty.$$

And for $p_j \leq 1, j = 1, \dots, m$ we can also provide some results on duals of these analytic classes following the proof from [1]. And mixing last two scales (quazinorms) with Bergman class quazinorm we get another two new analytic function spaces in product domains, we fix these six new function spaces in polydisk for further research.

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