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Original article

On the action of Toeplits operators into new BMOA type spaces in the unit disk

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Abstract. We provide new sharp results on the action of Toeplitz operators from Triebel and Besov spaces to new BMOA-type function spaces on the unit disk. We modify little our previously known proofs. We show our theorems using standard estimates of complex function theory. The proof of sufficiency and neccesity parts of our theorems follow the same type arguments as in previous already known cases. In particular we use standart test function for the proof of neccesity part and the same line of arguments as in previous already known cases. Our theorems may have many applications in complex function theory. Previously such type theorems were known in less general classses of BMOA type in the unit disk.

Keywords: Toeplitz operators, Besov type spaces, analytic functions, unit disk, BMOA type spaces

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Научная статья

О действиях операторов Теплица в классах типа BMOA в единичном круге

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Аннотация. Мы приводим новые точные результаты об операторах типа Теплица действующих на пространствач типа BMOA в единичном круге. Операторы Теплица имеют большое число приложений в теории функций комплексного переменного. В классах типа BMOA операторы Теплица рассматривались в ряде работ зарубежных авторов. Мы продолжаем эти исследования. Используются в доказательстве стандартные оценки из теории функций комплексного переменного. Результаты базируются на методах предыдущих работ первого автора. В доказательстве необходимости используются стандартные тест функции и оценки из предыдущих работ автора. Работа может иметь приложения к задачам теории функций комплексного переменного.

Ключевые слова: операторы Теплица, классы типа Бесова, аналитические функции, классы типа BMOA, единичный круг

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Introduction

In this note we will extend our previously known sharp theorems on the action of Toeplits operators into BMOA type function spaces in the unit disk (we consider $s \geq 1$ here). More precisely we provide new sharp results on the action of Toeplits operators from mixed norm analytic function spaces into new BMOA type classes in the unit disk. For that reason we modify the previously known proof, provided earlier by first author in classical function spaces.

Proofs of our sharp results and proofs of [1] are based mainly on similar type ideas.

We introduce new BMOA type spaces in the unit disk as follows.

$$BMOA_{s,q}(U) = \left\{ f \in H^s(U) : ||f||_{BMOA_{s,q}} = \right.$$

$$\left. = \sup_{z \in U} \left(\int_T \frac{|f(\xi) - f(z)|^s}{|1 - \xi \bar{z}|^q} dm(\xi)(1 - |z|^2) \right)^{1/s},\ 0 < q < \infty,\ 1 \leq s < \infty \right\};$$

$$BMOA_s^p(U) = \left\{ f \in H^s(U) : ||f||_{BMOA_s^p} = \right.$$

$$\left. = \sup_{z \in U} \left(\int_T \frac{|f(\xi) - f(z)|^s}{|1 - \xi \bar{z}|^2} dm(\xi)(1 - |z|^2)^p \right)^{1/s},\ 0 < p < \infty,\ 1 \leq s < \infty \right\}$$

(see definitions of all objects below). It is easy to see that in particular values of parameters quazinorms of these analytic spaces in the unit disk coincide with the so -called Garsia norm in BMOA (see [2], [3]-[6]).

The intention of this short paper to show new sharp results on the action of T_φ Toeplitz operators in some new BMOA type spaces in the unit disk. We provide a necessary and sufficient condition on the symbol of T_φ operator. Note such type results have various applications. Various results on BMOA type function spaces can be seen in papers [2], [3]–[6]. Various results on the action of T_φ Toeplitz operators can been seen in recent papers [2], [3], [6], [7] on various new and classical analytic function spaces in the unit disk. We refer to [2], [3], [6] and [8] for some applications of such type results in analytic function spaces.

Let further $U = \{z \in C, |z| < 1\}$ or D be the unit disk on a complex plane C, T be the unit circle on C. Let also further $I = (0,1)$. Let further H(U) be the space of all analytic functions in U .

In this paper we as usual denote by $\mathcal{D}^\alpha$ for any real α the fractional derivative of analytic f function in the unit disk,

$$\mathcal{D}^\alpha f(z) = \sum_{k=0}^{\infty}(k+1)^\alpha a_k z^k,\ z \in U$$

$$\text{for any analytic } f \text{ function, } f(z) = \sum_{k=0}^{\infty} a_k z^k,\ \alpha > -1,\ \alpha \in \mathbb{R} \quad (see[4]).$$

Note if $f \in H(U)$ then for any $s \in \mathbb{R}$, $D^s f \in H(U)$.

The Hardy spaces, denoted by $H^p(U)$ $(0 < p \leq \infty)$, are defined as usual (see [9]) by

$$H^p(U) = \left\{ f \in H(U)\colon \sup_{0<r<1} M_p(f,r) < \infty \right\},$$

where

$$M_p^p(f,r) = \int_T |f(r\xi)|^p\, dm_1(\xi), \quad M_\infty(f,r) = \max_{\xi \in T} |f(r\xi)|\,, r \in (0,1), f \in H(U).$$

For $\alpha > -1, 0 < p < \infty$, recall that the weighted Bergman space $A^p_\alpha(U)$ consists of all holomorphic functions on the unit disk satisfying the condition

$$\|f\|^p_{A^p_\alpha} = \int_U |f(z)|^p\,(1-|z|^2)^\alpha dm_2(z) < \infty \ \text{ (see [4, 6, 7, 9])}.$$

Let further $H(U)$ be the space of all analytic functions in U. Let further also (see [4], [4])

$$F^{p,q}_\alpha(U) = \left\{ f \in H(U) : \|f\|^p_{F^{p,q}_\alpha} = \int_T \left(\int_I |D^m f(r\xi)|^q (1-r)^{(m-\alpha)q-1} dr \right)^{\frac{p}{q}} d\xi < \infty \right\},$$

where $0 < p,\ q < \infty$, $m > \alpha, \alpha \in \mathbb{R}$, be the holomorphic Lizorkin-Triebel space, (see, for example, [4], [5]).

Let

$$F^{p,q}_{\alpha,k}(U) = \left\{ f \in H(U) : \|D^k f\|_{F^{p,q}_\alpha} < \infty \right\}, 0 < p, q, \alpha < \infty, k \in \mathbb{N}.$$

Note that we can easily show $F^{p,q}_\alpha$ general mixed norm analytic function spaces in the unit disk are Banach spaces for all values of p and q, if $\min(p,q) > 1$ and they are complete metric spaces for all other values of p and q.

Note (see [2], [4], [5], [7]) for particular case $p = q$ we have Bergman classical class, for $q = 2$ we have so-called Hardy-Lizorkin space H^p_β for some β that is, $H^p_\beta = \{f \in H(U) : D^\beta f \in H^p\}$, $0 < p \leq \infty$, $\beta > 0$, where D^β is a fractional derivative of analytic f function in U. Note (see definitions bellow) for this particular cases the action of $T\varphi$ classical Toeplitz operator is well-studied in unit disk, unit ball, unit polydisk and unit disk.

Various sharp results on action of Teoplitz and other operators can be seen in papers of various authors in various functional spaces in the unit ball, polydisk and unit disk. We mention, for example, the following papers [6] and [8], where such type sharp results can be seen for various cases of $F^{p,q}_\alpha$ spaces namely in Bergman type and in Hardy type spaces in the unit ball, polydisk and in the unit disk. We also note similar type results in for particular values of parameters are well-known also in other domains (see, for example, [6]).

Such type sharp result on boundedness of Toeplitz operators also have various applications (see, for example [6], [8]).

We remind the reader the standard definition of Toeplitz T_h operators in the unit disk.

Let $h \in L^1(T)$. Then we define Toeplitz T_h operator as an integral operator

$$(T_h f)(z) = \frac{1}{(2\pi)} \int_T \frac{f(\xi)h(\xi)}{(1-\bar{\xi}z)} dm(\xi), \quad z \in U.$$

We stress that behavior of the operators in the unit polydisk is substantially different from the action of $T\varphi$ operators in the unit ball in $\mathbb{C}^n$ (see [8] for example). Our intention to set criteria for the action of Toeplitz T_φ operators from $F^{p,q}_{\alpha,k}(U)$ into BMOA type spaces in the unit disk, under the assumption that φ is holomorphic, $\varphi \in H(U)$ (with some restriction on symbol of Toeplitz operator).

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

We pay special attention to places where different arguments from those we see in [1] are needed.

Lemma 1.

$$\textit{Let } R \in (0,1);\ w \in D;\ \phi \in H^\infty(D),\ s \leq 1,$$

$$\tau f_R(w) = F_R(w) = F(Rw) =$$

$$\frac{1}{2\pi i} \int\limits_\tau \frac{\overline{\phi(t)} f(t)}{1-\bar{t}Rw} dm(t),\ R \in I,\ w \in D.$$

Then

$$\left(\int\limits_T |F(\xi) - F(w)|^s \, \frac{(1-|w|)}{|1-\overline{w}\xi|^q} \, dm(\xi) \right)^{\frac{1}{S}} \leq$$

$$\leq c(||\phi||_\infty) \int\limits_T \int\limits_D \frac{|D^k f(z)|^s (1-|z|)^{ks+s-2}(1-|w|) dm(\xi)}{|1-z\bar{\xi}|^s |1-\overline{w}z|^s |1-\overline{w}\xi|^{q-s}} dm_2(z) \leq$$

$$\leq c(||\phi||_\infty) \left(\int\limits_D \frac{|D^k f(z)|^s (1-|z|)^{ks+s-2}}{|1-\overline{w}z|^{q-1+s}} \dot{(}1-|w|) dm_2(z) \right)^{\frac{1}{S}} = \mathcal{J} ||\phi||_\infty$$

$$||\phi||_\infty = \sup_{z \in D} |\phi(z)|.$$

Put $b = q - 3 + s, b > -1$ *and we have* $a = ks + 1 - q$ *. From here we have*

$$\mathcal{J}^s \leq c\Bigg(\Bigg(\sup_{r\in(0;1)}\Bigg)\Bigg(\int\limits_T |D^k\, f(r\xi)|^s d\xi\Bigg)(1-r)^q\Bigg) \times \Bigg(\int\limits_0^1 \frac{(1-r)^b\, dr}{(1-rR)^{q-1+s}}\Bigg)(1-R);$$

$$q - 1 + s > b + 1 \Leftrightarrow q + s > b + 2.$$

Theorem 1. *Let* $s < 1,\ 2 - s < q < 1 + s$*. Then* (T_ϕ) *is acting from* $S_{q,s}$ *:*

$$||f||_{S_{q,s}} = \Bigg(\sup_{r\in(0,1)}\Bigg) M_s(D^k\, f, r) \cdot (1-r)^{\frac{ks+1-q}{s}};$$

to $BMOA_{s,q}$*, if and only if* $\phi \in H^\infty$.

Proof. (of theorem 1) The sufficiency was provided above. Let us show the necessity part

$$Let\ f_r(z) = \frac{(1-r)^\gamma}{1-rz},\ \gamma > \gamma_0,\ r \in (0,1),\ z \in D.$$

We have

$$||T_\phi f||_{BMOA_{s,q}} \leq c||f||_{S_{q,s}},$$

then

$$||f||_{S_{q,s}} \leq c(1-r)^{\tau_0}$$

and (see [1])

$$||T_\phi f||_{BMOA_{s,q}} \geq c(1-r)^{\tau_0}|\phi(z)|, \quad \tau_0 = \gamma + \frac{2-s-q}{s}.$$

See ([]) . Now following arguments from (see [1]) we have

$$|\phi(z)| \leq c,\ so\, \phi \in H^\infty(D).$$

Lemma 1. *(see [1])*

$$det R \in (0,1),\ w \in D, \phi \in H^\infty(D),\ \ s \leq 1,$$

$$(Tf_R)(w) = F_R(w) = F(Rw) =$$

$$= \frac{1}{2\pi i}\int\limits_T \frac{f(t)\overline{\phi(t)}}{1-\overline{t}Rw}dt,\ R \in I,\ w \in D.$$

Then

$$||F||_{BMOA_s^q} = \Bigg(\sup_{z\in D}\Bigg)\Bigg(\int\limits_T \frac{|F(\xi) - F(z)|^s\, dm(\xi)}{|1-\overline{\xi}z|^2}(1-|z|)^q\Bigg)^{\frac{1}{S}} \leq$$

$$\leq c\int\limits_D |D^k f(r\xi)|^s \frac{(1-|z|)^{ks+s-2}}{|1-z\overline{w}|^{2s}}(1-|w|)^{q-1+s} dm_2(z) \cdot ||\phi||_{H^\infty} = I(f).$$

From last estimate we have

$$I \leq (a = ks + q - 2,\ b = s - q > -1) \leq$$

$$\leq \sup_{0<r<1}\left(\int\limits_T |D^k f(r\xi)|^s d\xi\right)\cdot(1-r)^a\cdot\int\limits_0^1 \frac{(1-|w|)^b}{(1-|z||w|)^{2s}}d(|z|)\cdot(1-|w|)^{q-1+s}\leq cA(f).$$

Theorem 2. *Let $S\in(0,\frac{1}{2})$, $1-s<q<1+s$. Then T_ϕ is bounded.*
From $\widetilde{S}_{s,q}$

$$||f||_{\widetilde{S}_{s,q}}=\left(\sup_{0<r<1}\right)\left(\int\limits_T |D^k f(r\xi)|^s d\xi\cdot(1-r)^{ks+q-2}\right)^{\frac{1}{s}}$$

to $BMOA_s^q$ if and only if $\phi\in H^\infty$.

Proof. (of theorem 2). The sufficiency part was provided above. Let us show the necessity

$$Let\ f_r(z)=\frac{(1-r)^\gamma}{(1-rz)};\ \gamma>\gamma_0,\ r\in(0,1),\ z\in D.$$

We have

$$||T_\phi f||_{BMOA_s^q}\leq c||f||_{\widetilde{S}_{q,s}}.$$

Then $||f||_{\widetilde{S}_{q,s}}\leq c(1-r)^{\tau_0}$, $\tau_0=\gamma+\frac{q-1-s}{s}$ and (see [1]) and

$$||T_\phi f||_{BMOA_s^q}\geq c(1-r)^{\tau_0}|\phi(z)|;$$

following arguments from [1] $|\phi(z)|\leq$ const, $\phi\in H^\infty(D)$.
By lemma 1

$$||T_\phi f||_{BMOA_{s,q}}\leq$$

$$\leq c\int\limits_T\int\limits_D \frac{|D^k f(z)|^s(1-|z|)^{ks+s-2}(1-|w|)}{|1-z\overline{\xi}|^s|1-\overline{w}\xi|^{q-s}|1-z\overline{w}|^s}dm_2(z)\cdot||\phi||_\infty\leq$$

$$\leq c||\phi||_\infty\left(\sup_{z\in D}|D^k f(z)|^s(1-|z|)^{ks+2-q}\right),$$

$$\int\limits_T\int\limits_D \frac{(1-|z|)^{s+q-4}(1-|w|)dm(\xi)dm_2(z)}{|1-z\xi|^s|1-\overline{w}z|^s|1-w\xi|^{q-s}}\leq \text{const.}\cdot I_2,$$

$s<1,\ q>1,\ q>(3-s),\ q-s>1$

$$I_2\leq c(1-|w|)^{-(2+s-q)}\int\limits_T \frac{d\xi}{|1-\overline{w}\xi|^{q-s}}(1-|w|)\leq \text{const}.$$

Using that

$$\int\limits_D \frac{(1-|z|)^{s+q-4}dm_2(z)}{|1-z\overline{\xi}|^s|1-w\overline{z}|^s}\leq c(1-|w|)^{-(2+s-q)}(see[1]).$$

$s-(s+q-4)<2,\ 2+s-q>0,\Leftrightarrow q>2,\ q<s+2.$

So we arrive at theorem 3.

Theorem 3. *The (T_ϕ) is acting as a bounded operator from $A_{s,q}^\infty$*

$$||f||_{A_{s,q}^\infty}=\left(\sup_{z\in D}\right)|D^k f(z)|(1-|z|)^{\frac{ks+2-q}{s}}<\infty$$

to $BMOA_{s,q}$ if and only if $\phi \in H^\infty(D)$, $s \in (\frac{1}{2}, 1)$ $q \in (3-s, s+2)$.

Proof. Sufficiency was provided above.
We show the reverse now

$$Let\ f_r(z) = \frac{(1-r)^\gamma}{1-rz},\ \gamma > \gamma_0;\ r \in (0,1);\ z \in D.$$

We have
$||T_\phi f||_{BMOA_{s,q}} \leq c||f||_{A^\infty_{s,q}}$, then $||f||_{A^\infty_{s,q}} \leq c(1-r)^{\tau_0}$ and (see [1])

$$||T_\phi f||_{BMOA_{s,q}} \geq c(1-r)^{\tau_0}|\phi(r)|\ (see[11]),\ \tau_0 = \gamma + \frac{2-s-q}{s}.$$

Following arguments from [1] we have $|\phi(z)| \leq c$, so $\phi \in H^\infty(D)$.

Using arguments of [1]

$$||T_\phi f||_{BMOA^q_s} \leq (s \leq 1, s > q)$$

$$\leq \int_T \int_D \frac{|D^k f(z)|^s (1-|z|)^{ks+s-2}(1-|w|)^q}{|1-\overline{w}z|^s|1-z\overline{\xi}|^s|1-\overline{w}\xi|^{2-s}} dm(\xi)dm_2(z) \leq$$

$$\leq c \sup_{t\in D} |D^k f(z)|^s (1-|z|)^{ks+q-1}\cdot$$

$$\cdot \int_T \int_D \frac{(1-|z|)^{s-q-1}(1-|w|)^q dm(\xi)dm_2(z)}{|1-\overline{w}z|^s|1-z\overline{\xi}|^s|1-\overline{w}\xi|^{2-s}} = c(I_1)\cdot(I_2).$$

Since $q < 1; s > 1-q; 2-s > 1$, $I_2 \leq c(1-|w|)^0 = \text{const}$. Since $\int_T \frac{d\xi(1-|w|)^q}{|1-w\xi|^{2-s}} \cdot (1-|w|)^{-q-s+1} \leq \text{const}$.

Since

$$\int_D \frac{(1-|z|)^{s-q-1}}{|1-\overline{w}z|^s|1-z\overline{\xi}|^s} dm_2(z) \leq c(1-|w|)^{-q-s+1};\ w \in D(see[1]).$$

We arrive at theorem 4.

Theorem 4. *Let $s \in (\frac{1}{2}, 1)$; $q \in (1-s, s)$.*
Then T_ϕ is a bounded operator from $\widetilde{A}^\infty_{s,q}$:

$$||f||_{\widetilde{A}^\infty_{s,q}} = \sup_{z\in D} |D^k f(z)|(1-|z|)^{\frac{ks+q-1}{s}} < \infty$$

to $BMOA^q_s$ if and only if $\phi \in H^\infty(D)$.

Proof. Proof of sufficiency was provided above.
Proof of reverse part

$$Let\ f_r(z) = \frac{(1-r)}{1-rz},\ \gamma > \gamma_0,\ r \in (0,1),\ z \in D.$$

We have

$$||T_\phi f||_{BMOA^q_s} \leq c||f||_{\widetilde{A}^\infty_{s,q}},$$

then $||f||_{A^\infty_{s,q}} \leq c(1-r)^{\tau_0}$; and see ([1]) $||T_\phi f||_{BMOA^q_s} \geq (1-r)^{\tau_0}|\phi(r)|$ (see [1])

$$\tau_0 = \gamma + \frac{q-1-s}{s}$$

Following arguments from [1] $\phi(z)| \leq c,\ z \in D$.

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